Constitutive Equations and the Correspondence Principle for the Dynamics of Gas Lubricated Triboelements

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A new method for characterizing the dynamic behavior of gas films in triboelements is developed. The new method is based on an expansion of the step jump method. In this method, the dynamic character of the gas film is preserved in the form of its force response to a step jump stimulus. Transforming this step response into the frequency domain yields the frequency dependent stiffness and damping properties of the gas film. By approximating the step response with analytic constitutive models and using the elastic-gas film correspondence principle, it is possible to determine the system characteristic equation in analytic form and to find closed-form solutions for stability, transient and forced responses. Requirements are given for choosing constitutive models that comply with the second law of thermodynamics. The new analytical solution method offers a significant time savings compared to direct numerical methods, and it is much more conducive to parametric studies.

Introduction

The effect of gas films in gas bearing systems was the subject of much interest in the sixties with the increasing popularity of gas turbine engines and rotor bearing systems. Much of the research was devoted to developing design schemes for bearing systems and finding methods by which the stability of these systems might be predicted. Including the gas film effect in the dynamic analysis of gas lubricated tribological elements is very difficult, even for simple geometries, because of the nonlinear nature of the gas film governing equation. The gas film, like other elements of the tribological mechanism, is a medium capable of storing and dissipating energy. Its stiffness and damping properties are both transient and frequency dependent. The problem with the analysis is made much more difficult because the boundaries of the gas film are themselves coupled to other dynamic equations. This coupling requires a simultaneous solution of both the gas film governing equation (i.e., the unsteady Reynolds equation) and the equations of motion for the triboelement.

A closed-form solution of the unsteady Reynolds equation for compressible flow is impossible, so numerical techniques have been employed from the beginning to help analyze the dynamics of gas lubricated triboelements. One important advantage of the full numerical simulation technique is that it gives a large amount of detailed information about the motion of each element in the mechanism but at the expense of high computing times. Another key advantage is that the method is capable of including the nonlinear terms in the equations. Castelli and McCabe (1967) and Tang (1971) were some of the earliest researchers to use this method, and now it is almost universal. However, every change in geometry, system parameters or initial conditions necessitates another complete solution, thus rendering this method inefficient for parametric studies or design.

A linear analysis by the perturbation method can provide a frequency domain solution of the Reynolds equation coupled with the equations of motion. Such a method was first used by Ono (1975) to characterize the frequency dependent dynamic properties of gas films in magnetic recording devices. Later, it was modified to yield the time transient behavior (Ono et al., 1979), and it has since been employed by several researchers (Tanaka et al., 1989 and Smith and Iwan, 1991).

Another technique was developed by Elrod et al. (1967) to help determine the stability of gas lubricated systems. This technique, called the step jump method, characterizes the stiffness of the gas film by its response to a step disturbance in each degree of freedom. These step responses are then converted into analytical functions by approximating them with a series of Laguerre polynomials. Since the dynamic character of the gas film is in analytic form, the method lends itself readily to a parametric analysis. The technique has been adopted by several researchers (e.g., Shapiro and Colsher, 1970; Kazimierski and Jarzecki, 1979; Etsion and Green, 1981; and Sela and Blech, 1991), and the method detailed by Elrod et al. (1967) has been reproduced in the textbook by Gross (1980). However, Miller and Green (1997) have shown that using Laguerre polynomials to characterize the gas film response can violate the second law of thermodynamics.

The advantages of the direct numerical solution method are so important that a full dynamic analysis of a potential design is left incomplete without at least some verification by numerical simulation. Nevertheless, there is some motivation for a linearized analysis by the perturbation method or the step jump method. The full numerical simulation does not provide much insight into the dynamic characteristics of the gas film or mechanism. However, the perturbation method gives the frequency response of all the elements in the mechanism as well as the gas film. Also, both the perturbation and step jump methods are more suited for a parametric study than the direct numerical method.

Although the perturbation method and the step jump method are both linearized techniques, there are two fundamental differences between the two methods. First, the perturbation method solves equations that are linearized about the equilibrium pressure and film thickness. In the step jump method, however, the Reynolds equation remains nonlinear in pressure. Second, both methods are readily used with stability tests such as the Routh-

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Hurwitz stability criterion, but the step jump method has the advantage of converting the step response into analytical format. Note that conflicting statements on the accuracy of the perturbation method are present in the literature. Some researchers (Ono, 1975 and Tanaka et al., 1989) suggest that the gas film in slider bearings can exhibit negative damping, but this phenomenon is physically infeasible. For example, Hayashi et al. (1990) showed a slider bearing in which negative damping in the gas film is predicted by the perturbation method, but a conventional, direct numerical analysis proved otherwise. In the analysis method presented here, significant attention is placed on creating constitutive models that comply with the second law of thermodynamics (see Miller and Green, 1997). Therefore, the problem of predicting negative damping will be avoided.

Based on the previous discussion, there is a need for an analysis technique that removes the requirement to solve the equations of motion and the Reynolds equation simultaneously using a numerical procedure. This need provides motivation to develop an analytic model for representing the gas film that embodies its distinctive stiffness properties. If the gas film force response can reasonably be assumed linear about equilibrium, then this task can be accomplished using the step jump method. In this work, three new constitutive models for the gas film that unconditionally satisfy the second law of thermodynamics are formed by approximating the step response with a Prony series, a series of complementary error functions, and a series of modified Bessel functions of the first kind of order zero. The constitutive law, then, acts as a kernel of the solution of the Reynolds equation. Using the new analytic constitutive models to represent the dynamic character of the gas film, the analysis continues analytically giving closed-form solutions.

Thrust Slider Bearing

In practical gas lubricated triboelement problems, instability may be caused by mechanisms other than the gas film (for example, gyroscopic effects in mechanical face seals). These types of problems are, therefore, unsuitable for use in establishing the principles developed here. An application that is unconditionally stable is needed. Consequently, all of the theory and examples presented here will relate to a generic gas lubricated, thrust slider bearing of infinite width (into the page) shown in

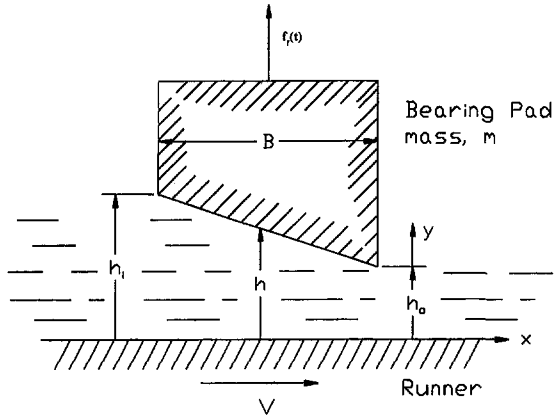

Fig. 1 Schematic of thrust slider bearing

Fig. 1. In this bearing, the gas film offers the only mechanism of dissipating energy, and, therefore, it is unconditionally stable (Miller and Green, 1997). The bearing pad is allowed to move only in the y-direction, and the bearing, therefore, has only one degree of freedom. The y-coordinate is measured from the equilibrium position where the net gas film force balances the external forces on the bearing pad. Ambient pressure conditions are applied at the inlet and outlet regions of the bearing. For the given geometry, the minimum film thickness is located at the trailing edge of the bearing pad and is designated by h_o. An arbitrary forcing function is designated by $f_f(t)$.

For this tribological application, the gas film is governed by the isothermal form of the unsteady Reynolds equation for compressible gases. To account for gas rarefaction effects, the Reynolds equation is modified to include second-order slip effects (Hsia and Domoto, 1983):

$$\frac{\partial}{\partial x}\left[ph^3 \frac{\partial p}{\partial x}\left(1 + 6\frac{K_n}{ph} + 6\frac{K_n^2}{(ph)^2}\right)\right] = \Lambda\left[\frac{\partial(ph)}{\partial x} + \frac{\partial(ph)}{\partial t}\right] \quad (1)$$

Nomenclature

A_n = nth coefficient
B = slider bearing length in x-direction
$f^*(t)$ = gas film force response
$f(t)$ = nondimensional gas film force, $f^*(t)/P_a(B\cdot 1)$
f_f = nondimensional external forcing function
f_n = net nondimensional gas film force
f_t = total nondimensional gas film force
$F(s)$ = Laplace transform of $f(t)$
h^* = film thickness
$h^*_{o,\mathrm{eq}}$ = minimum film thickness at equilibrium
h = nondimensional film thickness, $h^*/h^*_{o,\mathrm{eq}}$
j = imaginary unit, $(-1)^{-1/2}$
$k^*(t)$ = gas film step response, $\Delta f^*(t)/\Delta y^*$
$k(t)$ = nondimensional gas film step response, $k^*(t)h^*_{o,\mathrm{eq}}/P_a(B\cdot 1)$
k_g = pseudo-linear spring modulus
K_n = Knudsen number, $\lambda/h^*_{o,\mathrm{eq}}$
$K'(\omega)$ = storage modulus
$K''(\omega)$ = loss modulus
$K(s)$ = Laplace transform of $k(t)$
m^* = mass of the bearing pad per unit width
m = nondimensional mass, $m^*V^2h^*_{0,\mathrm{eq}}/4B^3P_a$
N = number of terms in step response approximation
p^* = pressure
p = nondimensional pressure, p^*/P_a
P_a = ambient pressure
s = Laplace variable
t^* = time
t = nondimensional time, $Vt^*/2B$
x^* = general coordinate
x = nondimensional variable, x^*/B
y^* = general coordinate
y = nondimensional displacement variable, $y^*/h^*_{o,\mathrm{eq}}$
Δy = step jump displacement
$Y(s)$ = Laplace transform of $y(t)$
V = velocity in x-direction
$Z(s)$ = impedance
$Z(\omega)$ = complex impedance
α_n = nth exponential attenuation factor
λ = mean free path of gas at ambient conditions
Λ = nondimensional compressibility number, $6\mu VB/P_a(h^*_{o,\mathrm{eq}})^2$
μ = gas absolute viscosity
ω^* = frequency
ω = nondimensional frequency variable, $2B\omega^*/V$

Subscripts and Superscripts

eq = equilibrium state
f = forcing
n = net; indices
o = outlet region
t = total
$*$ = dimensional variable
∞ = steady state; long time

Table 1 Common bearing parameters for cases 1–4

Bearing length	$B = .004$ m
Velocity	$V = 0.56$ m/s
Bearing pad mass per unit width	$m^* = 0.16$ kg/m
Ambient Pressure	$P_a = (10)^5$ N/m^2
Gas viscosity—air, helium	$\mu = 1.86\ (10)^{-5}$ Ns/m^2
Mean free path—air	$\lambda = 0.064\ \mu$m at ambient conditions
Mean free path—helium	$\lambda = 0.186\ \mu$m at ambient conditions

where

$$p = \frac{p^*}{P_a} \quad h = \frac{h^*}{h^*_{o,\text{eq}}} \quad x = \frac{x^*}{B} \quad t = \frac{V}{2B} t^*$$

$$K_n = \frac{\lambda}{h^*_{o,\text{eq}}} \quad \Lambda = \frac{6\mu V B}{P_a (h^*_{o,\text{eq}})^2} \qquad (2)$$

The effect of gas rarefaction is measured by the Knudsen number, K_n, which is the ratio of the mean free path of the gas at ambient conditions to the minimum film thickness. The mean free path of the gas depends on the type of gas used. Since air and helium are typically used in magnetic disk storage applications, they will be considered as lubricating gases in this work.

In this work, four test cases corresponding to the bearing in Fig. 1 will be examined. Cases 1 and 2 are lubricated with air, and cases 3 and 4 are lubricated with helium. The parameters for cases 1 and 2 are chosen to simulate one bearing operating at two different flying heights. It is assumed that these two cases have the same load bearing capacity, where case 1 is considered the reference case for determining this value. Then for case 2, the inlet to outlet height ratio is adjusted until the load bearing capacity equals that of case 1. The parameters for cases 3 and 4 were chosen similarly. The parameters given in Table 1 are common to both bearings and, therefore, are applicable to all four cases. Table 2 specifies the bearing geometry and other parameters for each individual case. Although the dimensional bearing pad mass, m^*, is the same for all four cases, the nondimensional bearing pad mass, m, changes because of the nondimensionalizing process (see nomenclature).

Gas Film Dynamic Characterization

An analytic model for the dynamic stiffness and damping properties of gas films can be formulated by the step jump method. The method expresses a mathematical relationship between the displacement in one degree of freedom and the net gas film force resulting from the transient pressure diffusion in the gas film. It is based on the assumption that the net gas film force is linear in response to successive, small step jumps in each degree of freedom. Consequently, the effects of a series of step jumps can be individually superposed using Duhamel's integral (Miller and Green, 1997):

$$f_n(t) = k(t)y(0) + \int_0^t \dot{y}(\tau)k(t-\tau)d\tau \qquad (3)$$

where $\dot{y} = dy/d\tau$ and $k(t)$ is the gas film step response. Note that $f_n(t)$ is considered positive when pointed away from the bearing pad and into the gas film. Equation (3) represents a force-displacement constitutive law. Transforming this equation into the Laplace domain yields a very useful relationship,

$$F_n(s) = sK(s)Y(s) \qquad (4)$$

where $F_n(s)$, $K(s)$, and $Y(s)$ are the Laplace transforms of $f(t)$, $k(t)$, and $y(t)$, respectively. This equation forms the elastic-gas film correspondence principle, which will be employed later.

The gas film step response represents the transient gas film stiffness, which is an approximation of the net change in force in response to an infinitesimally small change in film thickness. Because the force response is assumed linear about the equilibrium film thickness, the step response completely characterizes the dynamic stiffness and damping of the gas film. Therefore, the step response forms a kernel of the solution of the Reynolds equation. As a result, once $k(t)$ is obtained, it is unnecessary to solve the Reynolds equation during a dynamic analysis of the equations of motion for a gas lubricated triboelement.

The step response is calculated directly by numerical solution of the unsteady Reynolds equation. The procedure for calculating the response to a single, finite step jump is detailed by Miller and Green (1997). In that work, an approximation for the step response for a step jump in the positive y direction is defined by the following equation:

$$k_{\Delta y}(t) = -\frac{f_t(t) - f_{\text{eq}}}{\Delta y} \qquad (5)$$

At the onset of the step disturbance, the identity ph = constant is an equation of state corresponding to isothermal conditions. For a more accurate approximation, the process above is also repeated for steps of $\frac{1}{2}\Delta y$, $-\frac{1}{2}\Delta y$, and $-\Delta y$ and averaged together using Richardson's extrapolation (Chapra and Canale, 1985) giving

$$k(t) = \tfrac{1}{6}\{4[k_{1/2\Delta y}(t) + k_{-1/2\Delta y}(t)] - [k_{\Delta y}(t) + k_{-\Delta y}(t)]\} \qquad (6)$$

Using this procedure, the step response, $k(t)$, varied only slightly in the third significant decimal place for step jumps in the range of $\Delta y = 0.30$. The step response curves for cases 1–4 are shown in Fig. 2.

More insight into the dynamic character of the gas film can be seen by considering its properties represented in the Laplace and frequency domains. In the Laplace domain, the ratio of the net gas film force and the displacement is defined as the impedance, $Z(s)$. According to Eq. (4), the impedance is

$$Z(s) \equiv sK(s) = \frac{F_n(s)}{Y(s)} \qquad (7)$$

To get the frequency domain version, replace s with $j\omega$ to yield the complex impedance,

$$Z(\omega) \equiv j\omega K(j\omega) = K'(\omega) + jK''(\omega) \qquad (8)$$

The real part of the complex impedance, $K'(\omega)$, is the storage modulus, and the imaginary part, $K''(\omega)$, is the loss modulus. These terms can be obtained directly using the Fourier cosine and sine transforms (Miller, 1996),

Table 2 Individual bearing parameters for cases 1–4

Bearing	Case number	Gas type	$h^*_{o,\text{eq}}$	$h_{i,\text{eq}}/h_{o,\text{eq}}$	K_n	Λ	m
Bearing #1	Case 1	Air	0.5 μm	1.800	0.128	10.0	10^{-6}
	Case 2	Air	0.1 μm	1.335	0.640	250.0	$2\ (10)^{-7}$
Bearing #2	Case 3	Helium	0.5 μm	1.800	0.372	10.0	10^{-6}
	Case 4	Helium	0.1 μm	1.243	1.860	250.0	$2\ (10)^{-7}$

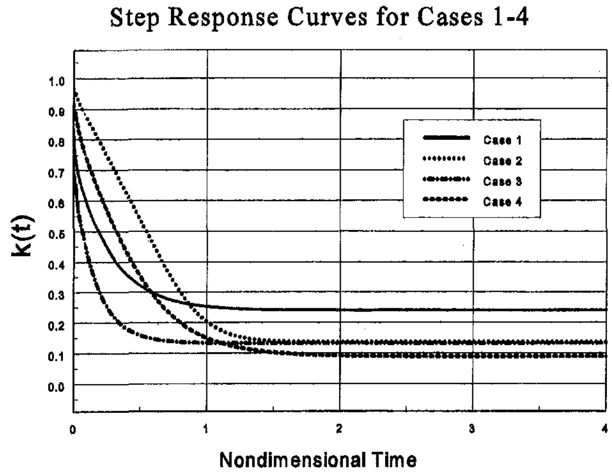

Fig. 2 Step response curves for cases 1–4

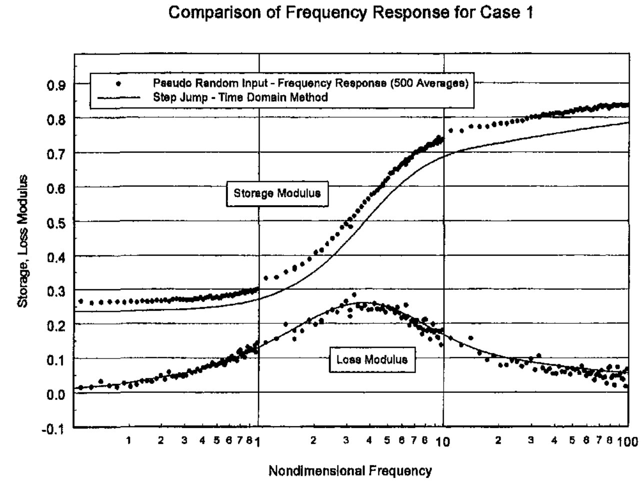

Fig. 4 Comparison of the frequency response for case 1 generated by a direct numerical method and by transforming the step response into the frequency domain

$$K'(\omega) = \int_0^\infty \dot{k}(t)\cos(\omega t)dt$$

$$K''(\omega) = -\int_0^\infty \dot{k}(t)\sin(\omega t)dt \quad (9)$$

The storage and loss moduli can be obtained either analytically, if $k(t)$ is known, or by numerical integration when the step response, $k(t)$, is given in tabular form as obtained from the step jump method. Using the above procedure, the storage and loss moduli for cases 1–4 are calculated and presented in Fig. 3. These curves show characteristics that are common to viscoelastic materials. The storage moduli begin at the rubbery modulus at low frequencies and proceed smoothly through the transition region to the glassy modulus at high frequencies. Likewise, the loss moduli start at zero at low frequencies and show a peak in the transition region and end up settling toward zero at high frequencies.

The curves generated by transforming the step response into the frequency domain are just approximations of the gas film complex impedance because the step jump method assumes that the product ph = constant at the onset of the step jump in displacement. To test the validity of using this procedure to approximate the gas film complex impedance, the results obtained by transforming the step responses into the frequency domain using Eq. (9) are compared to results obtained by a direct numerical method, which is discussed in detail in Appendix A.

The storage and loss moduli for case 1 calculated by both procedures are compared in Fig. 4. For this case, transforming $k(t)$ into the frequency domain produces a loss modulus that closely follows the curve obtained by the direct numerical method. However, for the storage modulus, the two curves are slightly offset, although the trend is still very much the same. It is suggested that the offset in the storage modulus is due to the ph = constant approximation at $t = 0$ in the step jump method. The small error introduced by this assumption shifts the high frequency asymptotes by a small percentage. As a result, only the storage modulus is affected by the approximation since the high frequency asymptote is zero for the loss modulus. For conciseness, the comparisons between the two methods for cases 2–4 are not shown here but can be found in Miller (1996). For each of these curves, the results are very similar to the results shown in Fig. 4.

The amount of computing time needed to obtain the step response and to transform it into the frequency domain is approximately 30 to 60 seconds on a 90 MHZ personal computer. However, the smallest time necessary to obtain the complex impedance using the direct numerical method is approximately 50 to 60 minutes on the same computer. Since the step jump method results represent the gas film very well and since the computing time is far less than for the direct numerical method, it is concluded that transforming the step response into the frequency domain is the preferable method of describing the gas film stiffness and loss properties.

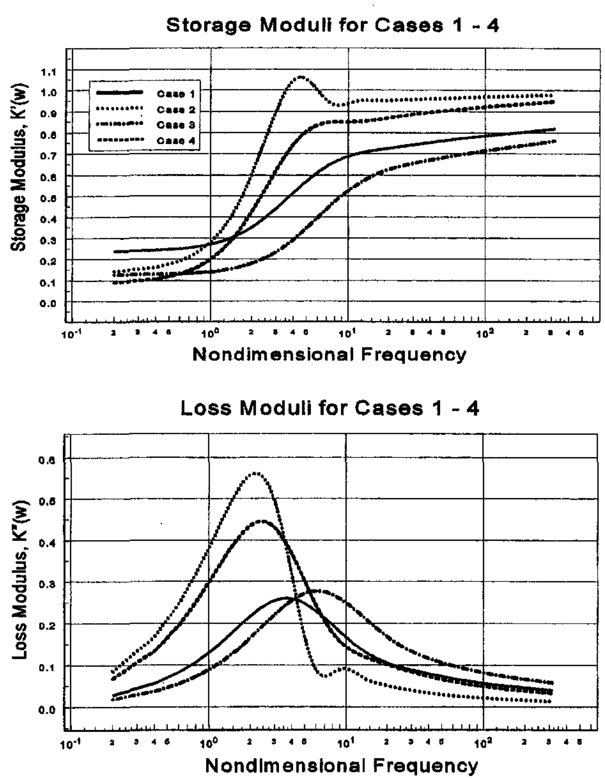

Fig. 3 Storage and loss moduli for cases 1–4

New Constitutive Models

Once the step response has been calculated by a numerical procedure, the analysis is made analytic by approximating the curve with an analytic function. The analytic approximations of the step response curves are called constitutive models, and these functions are subject to constraints. Functions used to approximate the step response must satisfy a criterion based on the second law of thermodynamics (Fabrizio and Morro, 1988 and Miller and Green, 1997). The criterion requires the loss modulus, $K''(\omega)$, which is defined above in Eq. (9), to be positive semi-definite over the range of frequencies from zero to infinity. Furthermore, it is important to choose functions that have representations in both the time and Laplace domains since much of the analysis procedure that uses the approximation is done in the Laplace domain.

In accordance with these two stipulations, three new constitutive models are formulated by approximating the step response with a Prony series, a series of complementary error functions, and a series of modified Bessel functions of the first kind of zero order. These models have previously been used to model the behavior of linear viscoelastic materials (Szumski, 1993). These particular analytic functions were chosen because they have both time and Laplace domain representations and satisfy the second law of thermodynamics for all sets of positive material constants. The time domain representations of the constitutive models are not often used directly. In most cases, the Laplace domain forms are used directly in analyses. The three constitutive models are given below.

1. Prony Series. The step response can be approximated with a series of decaying exponential functions,

$$k(t) = k(\infty) + \sum_{n=1}^{N} A_n e^{-\alpha_n t} \quad (10)$$

where α_n are the decay parameters and A are constant coefficients. For this series, the $sK(s)$ function is given by

$$sK(s) = k(\infty) + \sum_{n=1}^{N} A_n \frac{s}{s + \alpha_n} \quad (11)$$

2. Complementary Error Function Series. A series of complementary error functions is also used to approximate the step response,

$$k(t) = k(\infty) + \sum_{n=1}^{N} A_n e^{\alpha_n^2 t} \operatorname{erfc}(\alpha_n \sqrt{t}) \quad (12)$$

where erfc $(\cdot)$ is the complementary error function. The corresponding $sK(s)$ function is given by

$$sK(s) = k(\infty) + \sum_{n=1}^{N} A_n \frac{\sqrt{s}}{\sqrt{s} + \alpha_n} \quad (13)$$

3. Series of Modified Bessel Functions of the First Kind of Zero Order. The third approximating function is given below as

$$k(t) = k(\infty) + \sum_{n=1}^{N} A_n e^{-\alpha_n t} I_0(\alpha_n t) \quad (14)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of zero order. The appropriate $sK(s)$ function is given by

$$sK(s) = k(\infty) + \sum_{n=1}^{N} A_n \frac{\sqrt{s}}{\sqrt{s + 2\alpha_n}} \quad (15)$$

In all three of these models, A_n, α_n and $k(\infty)$ depend on the gas film and operating parameters. The values for A_n and α_n are determined using a curve fitting process either in the time domain with the step response or in the frequency domain with the complex impedance. Note again that the complex impedance, $Z(\omega)$, for each constitutive model is obtained by replacing s with $j\omega$ [see Eq. (8)]. In most instances, the curve fit on the storage and loss moduli gives a preferable fit since the dynamic characteristics of the gas film are better portrayed in the frequency domain representation of the step response than in the time domain representation.

The constitutive models represented by each of these three functions become an exact fit if the number of terms is allowed to approach infinity. However, a very large number of terms is impractical for computing purposes. The number of terms actually needed depends on the shape of the step response curve and the degree of accuracy wanted. This number can be chosen either by a trial and error method or by applying a criterion to measure the quality of fit. A trial-and-error procedure is used in this work. For the Prony series, typically a small number of terms, only two or three, provides an adequate fit for the step responses in both the time domain and the frequency domain. For the complementary error function and Bessel function approximations, one term constitutive models prove to be robust. As an alternative to the curve fitting procedure, it is also possible to use a simpler collocation method in which a collection of points may be used to emphasize significant regions in the time or frequency domain curves.

The curve-fit parameters for the four test cases are summarized in Table 3. These parameters are calculated by a least squares method based upon a multi-variable function minimizing process using a Nelder-Mead simplex algorithm. The function minimized in the algorithm is the norm of the difference between the data points obtained numerically and the constitutive equation.

The step response for case 1 is shown in the time domain in Fig. 5 along with the three approximate curves from the constitutive equations. Figure 6 shows the storage and loss moduli and their approximate curves for case 1 in the frequency domain. It is evident from the loss moduli that the three constitutive models are thermodynamically valid for the chosen set of material parameters. By observation, it appears that the Prony series fit approximates the actual data the closest, especially in the transition region from rubbery modulus to glassy modulus.

Dynamic Analysis Using the Constitutive Law

The dynamic contribution of the fluid layer in gas lubricated systems can be incorporated into the dynamic model using the elastic-gas film correspondence principle (Miller and Green, 1997). It is a direct result of the Laplace domain representation of the force-displacement relationship for gas films given in Eq. (4). According to the principle, a gas lubricated problem can be formulated by first modeling the gas film stiffness as if it were a spring with a constant spring modulus, k_g, and then appropriately developing the time-domain equations of motion. After transforming the equations into the Laplace domain, the transient stiffness property of the gas film is incorporated into the problem by replacing the pseudo stiffness, k_g, with $sK(s)$. The resulting equation is a Laplace domain version of the equations of motion, and it contains the transient nature, or the frequency dependence, of the gas film. This equation is the basis for the new solution technique discussed below.

When the equations of motion are attained in the Laplace domain, several solution options are available. At this point, it is possible to write any variable of interest in terms of analytic functions of the Laplace variable, s, and with that to obtain the time response by inverse Laplace transform. Also, it is possible to form a transfer function relating any two output-input variables of interest. Furthermore, the system characteristic equation can be written.

To illustrate these procedures, consider the general problem of forced vibration of the thrust slider bearing shown in Fig. 1. The equation of motion can be written in the time domain as,

$$m\ddot{y}(t) = f(t) + f_f(t) \quad (16)$$

The variable $f(t)$ is the net gas film force, and $f_f(t)$ is the forcing function. The elastic-gas film correspondence principle states that the gas film should be modeled as a pseudo linear spring, in which case $f(t) = -k_g y(t)$, giving

Table 3 Curve fit parameters for prony series, complementary error function series, and Bessel function series

		Prony series			Comp. error function			Bessel function		
Case #	$k(\infty)$	N	A_n	α_n	N	A_n	α_n	N	A_n	α_n
1	0.24070	3	$A_1 = 0.0462$ $A_2 = 0.0579$ $A_3 = 0.4965$	$\alpha_1 = 309.933$ $\alpha_2 = 56.261$ $\alpha_3 = 3.567$	1	$A_1 = 0.6214$	$\alpha_1 = 1.783$	1	$A_1 = 0.5296$	$\alpha_1 = 15.228$
2	0.13647	2	$A_1 = 0.0191$ $A_2 = 0.8306$	$\alpha_1 = 237.887$ $\alpha_2 = 1.438$	1	$A_1 = 0.8694$	$\alpha_1 = 0.583$	1	$A_1 = 0.8200$	$\alpha_1 = 5.416$
3	0.13226	2	$A_1 = 0.1017$ $A_2 = 0.5468$	$\alpha_1 = 149.414$ $\alpha_2 = 6.377$	1	$A_1 = 0.7021$	$\alpha_1 = 2.574$	1	$A_1 = 0.5861$	$\alpha_1 = 30.094$
4	0.08914	2	$A_1 = 0.0679$ $A_2 = 0.7984$	$\alpha_1 = 108.224$ $\alpha_2 = 2.136$	1	$A_1 = 0.8990$	$\alpha_1 = 1.033$	1	$A_1 = 0.8025$	$\alpha_1 = 7.823$

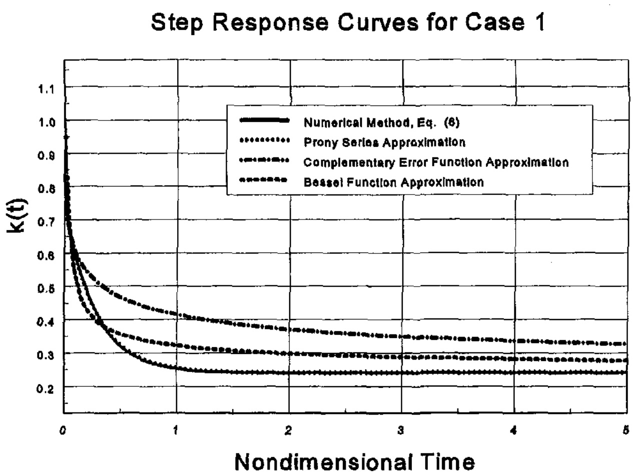

Fig. 5 Step response curves for case 1

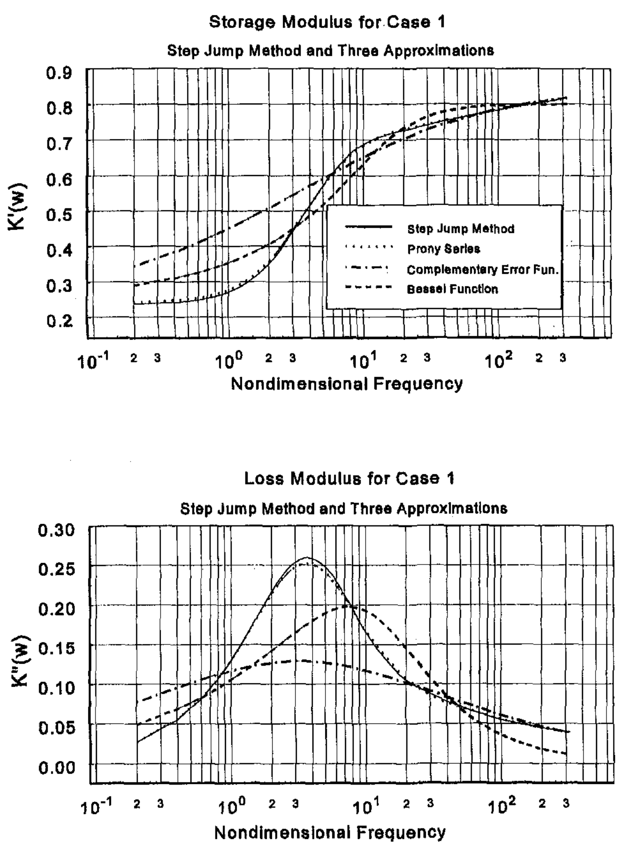

Fig. 6 Actual and approximate storage and loss moduli for case 1

$$m\ddot{y}(t) = -k_g y(t) + f_f(t) \tag{17}$$

Then, transforming the resulting equation into the Laplace domain yields

$$m[s^2Y(s) - sy(0) - \dot{y}(0)] = -k_g Y(s) + F_f(s) \tag{18}$$

According to the elastic-gas film correspondence principle, $sK(s)$ is substituted for k_g giving

$$m[s^2Y(s) - sy(0) - \dot{y}(0)] = -sK(s)Y(s) + F_f(s) \tag{19}$$

The terms $y(0)$ and $\dot{y}(0)$ are the initial displacement and velocity of the bearing pad, respectively. The characteristic equation is easily extracted,

$$ms^2 + sK(s) = 0 \tag{20}$$

The roots of this equation are the system eigenvalues.

Reconsidering the Laplace domain equation of motion for the problem at hand, a transfer function can be written relating the bearing pad displacement and the forcing function.

$$\frac{Y(s)}{F_f(s)} = \frac{1}{ms^2 + sK(s)} \tag{21}$$

Replacing s with $j\omega$ in this equation yields the frequency response,

$$\frac{Y(\omega)}{F_f(\omega)} = \frac{1}{K'(\omega) - m\omega^2 + jK''(\omega)} \tag{22}$$

An explicit expression for the complete bearing pad response can be written:

$$Y(s) = \frac{m[y(0)s + \dot{y}(0)]}{ms^2 + sK(s)} + \frac{F_f(s)}{ms^2 + sK(s)} \tag{23}$$

The time history of displacement for the bearing pad is now available by obtaining the inverse Laplace transform of this equation either analytically or numerically.

Certainly, the solutions just discussed are not the only ones possible with this technique. These are just a few representative examples. The important factor is that the gas film stiffness properties have been characterized and written in a compact form. This breakthrough introduces new analytic, Laplace and frequency domain solution techniques to the dynamic analysis of gas lubricated triboelements.

Results

To measure how effectively the new constitutive models approximate the gas film properties, the dynamic properties of the bearing system are found using the previously described technique and compared to results obtained using conventional numerical methods. The dynamic properties compared are the damped natural frequencies and the time history of displacement of the bearing pad. For the conventional numerical method, the time history of bearing pad response is found directly from the

Table 4 **Damped natural frequencies for cases 1–4 and the estimated values obtained using the new constitutive models (all values nondimensional)**

Case #	System damped natural frequency from DFT	Prony series approximation (relative error)	Complementary error function approx. (relative error)	Bessel function approximation (relative error)
1	906.32	914.63 (0.9%)	914.66 (0.9%)	877.57 (3.2%)
2	2221.80	2220.33 (0.07%)	2234.23 (0.6%)	2186.89 (1.6%)
3	877.63	882.17 (0.5%)	890.35 (1.4%)	847.19 (3.4%)
4	2194.20	2185.61 (0.4%)	2207.15 (0.6%)	2111.49 (3.8%)

output of a full numerical integration technique, and the damped natural frequencies are found by identifying the resonant peaks in the spectrum of the response time history.

The system damped natural frequencies found from the numerical simulation and from the new technique with the three new constitutive models are listed in Table 4. The estimated system eigenvalues are found by substituting the appropriate analytic approximation for $sK(s)$ into Eq. (20) and solving for the roots, or eigenvalues, of this characteristic equation with a numerical algorithm. The damped natural frequencies are the imaginary parts of these eigenvalues. The relative errors between the estimated and the actual system damped natural frequencies are shown in parentheses in Table 4. The results found using the new technique with all three constitutive models are in excellent agreement with those achieved from the full numerical simulation. The accuracy achieved by this new technique is not surprising since the curve fits for the storage and loss moduli for all four cases and with all three constitutive models are close in the frequency range of interest. If the bearing pad oscillated at a frequency that was in a range where the approximate storage and loss moduli did not match up well with the actual frequency response, then the predicted values would be less accurate. However, in special cases when it is known in advance that the system will be oscillating in a certain frequency band, it is possible to adjust the curve fitting algorithm to emphasize the frequency band of interest and to minimize the relative error in that range.

One important advantage of this analysis technique is that it can give a time history of a variable of interest. In this case, the time history of bearing pad displacement is found by performing an inverse Laplace transform the $Y(s)$ expression. For the displacement expression with the Prony series approximation for $sK(s)$, the inverse Laplace transform can be computed analytically, so a closed form solution to the bearing pad displacement can be formulated. Figure 7 shows a comparison of the time histories of bearing pad displacement for case 1 (with an initial nondimensional velocity of -50.0) calculated by a full numerical simulation of the coupled set of the Reynolds and the motion equations and by the analytic solution obtained from the new method with the Prony series constitutive model. The correlation between these curves shows that the predicted bearing pad displacement matches up well with the result from the numerical solution. This figure not only shows that the damped natural frequency is predicted to be very close to the full numerical solution, but also that the damping itself is estimated accurately since the decay envelopes of the two curves are very close. For the other two constitutive models, the inverse Laplace transform must be calculated numerically.

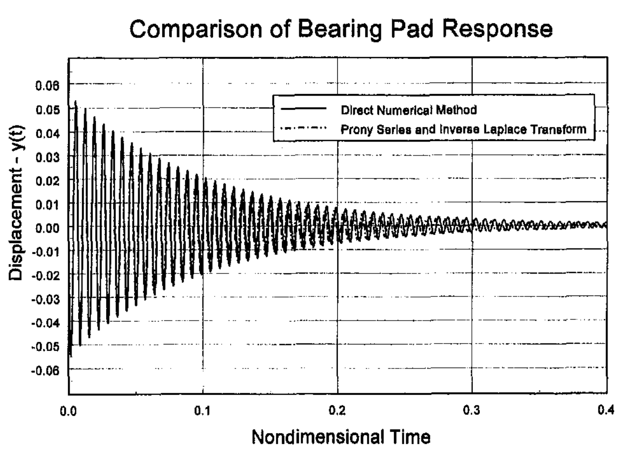

Fig. 7 Comparison of bearing pad response predicted using inverse Laplace transform and direct numerical method

Although an example is not presented here, the new technique can also be used to analyze problems with external forcing functions that can be represented in the Laplace domain. Some examples of common forcing functions that have analytic Laplace transforms are the impulse, step, ramp and sinusoidal functions. In this case, the characteristic equation is the same as for unforced problems. Then for the time domain solution, the technique for the forced problem is similar to that of the initial value problem in that an explicit expression for $Y(s)$ can be attained and the inverse Laplace transform can then be computed.

Conclusions

It has been shown in this work that transforming the step response into the frequency domain provides an accurate model of the gas film stiffness and damping properties. Using the step jump method also takes less computing time than generating the frequency response using a direct, numerical method. The computing time savings is approximately two orders of magnitude. Such savings are significant particularly when more realistic, multiple degrees of freedom systems are analyzed.

Also, a new solution procedure has been presented for analyzing the dynamics of gas lubricated triboelements. This new procedure represents the gas film analytically in the form of a constitutive model, thus removing the need to solve the Reynolds equation and the equations of motion simultaneously. The constitutive models must comply with the second law of thermodynamics. Even for complex tribosystems, once a thermodynamically admissible constitutive model is formed for the gas film, it can be incorporated into the dynamic analysis without fear of introducing false instabilities because of misrepresenting the gas film. Using the elastic-gas film correspondence principle, the solution of the complex problem of the triboelement dynamics coupled with the gas film is obtained analytically rather than numerically. As long as the gas film parameters remain the same, the system can be investigated to find the effect of a change in any system parameter, such as mass, support stiffness, forcing function, etc., with relative ease and quickness.

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APPENDIX A

Direct Numerical Procedure for Determining the Gas Film Frequency Response

A novel, direct numerical method is introduced here for determining the complex impedance, or frequency response, of the gas film. In this approach, a pseudo-random input displacement is applied to the bearing pad in the form of

$$y(t) = \sum_{i=1}^{M} Y_i \sin(\omega_i t) \tag{24}$$

Y_i is the relative amplitude of the i^{th} sinusoid. It is chosen to be a random number in a range of ±5 percent of the flying height. This amplitude range is much smaller than the 30 percent amplitude range in which the force response was found to be approximately linear. The M frequencies, ω_i, are chosen in even increments in a range that provides a suitable amount of information for frequencies below, through and above the transition region. The displacement form is chosen to be a summation of sine functions since they are infinitely differentiable and they yield the initial displacement to be zero (hence, the assumption ph = constant is moot). Also, the bearing pad velocity can be found in closed form,

$$\dot{y}(t) = \sum_{i=1}^{M} \omega_i Y_i \cos(\omega_i t) \tag{25}$$

The continuous forms of $y(t)$ and $\dot{y}(t)$ above comply with the need to have analytical representations of the film thickness and its time derivative as needed in the Reynolds equation during the time integration process.

Now, the problem requires calculating the resulting net gas film force (i.e., the total force minus the force at equilibrium) from a direct numerical solution of the unsteady Reynolds equation. The first step in the process is to determine the equilibrium state. Then, the displacement and velocity as specified in Eq. (24) and Eq. (25), respectively, are applied to the bearing pad, and the Reynolds equation is integrated forward in time. At each time increment, the net gas film force is calculated and stored along with the displacement. Next, the spectra of the displacement and net force response, $Y(\omega)$ and $F_n(\omega)$, respectively, are calculated digitally by a fast Fourier transform (FFT). Then, the gas film frequency response is defined as

$$Z(\omega) \equiv \frac{F_n(\omega)}{Y(\omega)} = K'(\omega) + jK''(\omega) \tag{26}$$

The spectra computed by the FFT are tabulated at discrete, evenly spaced frequencies in the range of interest. To make each ω_i of the input sinusoids correspond to one discrete frequency in the spectra, the number of terms in the input displacement, M, is chosen to equal half the number of points in the FFT.

The procedure described so far yields one set of M data points in the frequency response, and the result of this complete procedure is called a record. One record, however, inherently contains a significant amount of random error due to the pseudo-random nature of the displacement. To reduce the apparent random nature of the results, several records are computed by repeating the process above and averaged together. It is well known from random data analysis theory (Bendat and Piersol, 1986) that the random error decreases proportionally with the square root of the number of averages. However, the computing time increases linearly with the number of averages. Consequently, for this work 500 averages are chosen as a compromise between the desire to reduce the computing time and the amount of random error.