Exact Spectral Moments and Differentiability of the Weierstrass-Mandelbrot Fractal Function

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Fractal mathematics using the Weierstrass-Mandelbrot (WM) function has spread to many fields of science and engineering. One of these is the fractal characterization of rough surfaces, which has gained ample acceptance in the area of contact mechanics. That is, a single mathematical expression (the WM function) contains characteristics that mimic the appearance of roughness. Moreover, the "roughness" is "similar" across large dimension scales ranging from macro to nano. The field of contact mechanics is largely divided into two schools of thought: (1) the roughness of real surfaces is essentially random, for which stochastic treatment is appropriate, and (2) surface roughness can be reduced to fractal mathematics using fractal parameters. Under certain mathematical constraints, the WM function is either stochastic or deterministic. The latter has the appeal that it contains no randomness, so fractal mathematics may offer closed-form solutions. Spectral moments of rough surfaces still apply to both approaches, as these represent physical metrology properties of the surface standard deviation, slope, and curvature. In essence, spectral moments provide a means of data reduction so that other physical processes can subsequently be applied. It is well known, for example, that the contact model of rough surfaces, by Greenwood and Williamson (GW), depends on parameters that are direct outcomes of these moments. Despite the vast amount of publications on the WM function dedicated to surfaces, two papers stand out as originators, where the others mostly rework their results. These two papers, however, contain some omissions and approximations that may lead to gross errors in the estimation of the spectral moments. The current work revisits these papers and adds information, but departs in the mathematical treatment to derive exact expressions for the said moments. Moreover, it is said that the WM function is nondifferential. That is also revisited herein, as another approach to derive the spectral moments depends on such derivatives. First, the complete mathematical treatment of the WM function is made, then the spectral moments are derived to yield exact forms, and finally, examples are given where the physical meanings of the approximate and exact moments are discussed and their values are compared. Numerical procedures will be introduced for both, and the effectiveness of the computational effort is discussed. One numerical procedure is particularly effective for any digitized signal, whether that originates from analytical functions (e.g., WM) or real surface measurements. [DOI: 10.1115/1.4045452]

Keywords: Weierstrass-Mandelbrot fractal function, spectral moments, surface roughness, surface metrology, contact mechanics, metrology, contact mechanics, interface, microtribology, nanotribology, surface properties and characterization, surface roughness and asperities, surfaces

1 Introduction—Theoretical Background

Often in tribology, surface roughness is characterized by the spectral moments, m_0, m_2, and m_4, which are measures of the variance, slop, and curvature, respectively. They are completely sufficient to execute, for example, the Greenwood and Williamson (GW) contact model [1] under elastic conditions and other models under elastoplastic conditions. The work by McCool [2,3], for example, provides a complete mathematical procedure on how to convert two surfaces having two-dimensional orthotropic roughness into a single surface having a composite roughness described by a single set of m_0, m_2, and m_4.

Evidently, however, these moments are not just specific to modeling surface roughness in tribology, as they are central in the many fields of science and engineering that fall generally into the category of signal processing for which there is ample literature (see notably the classical texts by Bendat and Piersol [4–6]). Nontribological examples can range from the geomechanics of rough wall fracture [7] to signal processing performed on the output from the pulsed laser photoacoustic instrument monitoring crude oil in water [8], or to the analysis performed in an optical telescope [9]. Because of their general importance, the spectral moments are focal in this work.

In that framework, the work by Majumdar and Tien [10] postulates that the Weierstrass-Mandelbrot (WM) function can be "… used to simulate deterministically rough surfaces which exhibit statistical resemblance to real surfaces." They obtain spectral moments using the power spectrum derived by Berry and Lewis [11]. Hence, these two works are central herein. However, because the spectral moments as given in Ref. [10] are derived based on an "approximate" power spectrum that is given in Ref. [11], they can only be considered as "approximations." Their moments will be compared against the exact moments, which are derived herein.

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The Mathematica code can be obtained from the author upon request.

```
(* This code is provided by the author, I. Green, to complement the subject paper. You may use the code freely. *)
(* Exercise any or all of the cases listed in Table 1 in
 the said paper. This code will reproduce all of the results in there. *)
(* Missing from this code is the implementation of the spectral moments as found
 in Ref.[10] because they produce results that are grossly in error. *)
(* The errors are caused by the approximated power spectrum,
sometimes dubbed as "the continuous power spectrum density," as derive in Ref. [11].
  That spectrum had been used in Ref. [10] and, since then, in many other papers. The errors sometimes approach 100%!
  These errors put the "approximated/continuous power spectrum," and the so-called "power law," in question.  *)
```

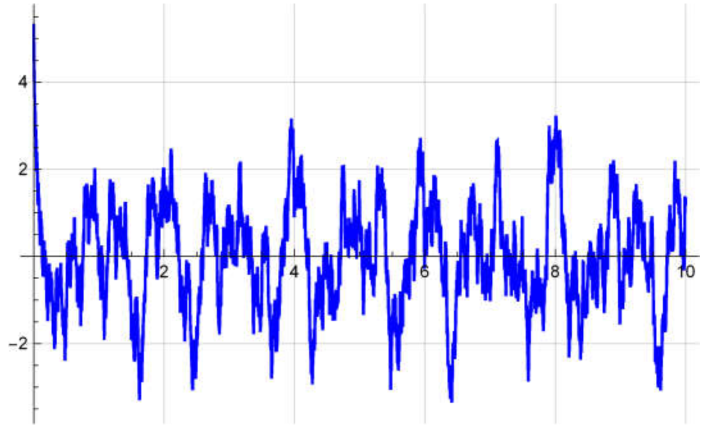
-----: DD = 1.5

-----: g = 1.5

-----: G = 1.

-----: q = 1

-----: n2 = 17



The EXACT Spectral Moments according to the IG paper (it is implied that LL−>Infinity);

These are entirely analytical, so there is NO CPU time consumed, at all!

-----: sm0 = m[0] = 1.49899

-----: sm2 = m[2] = 58 305.4

-----: sm4 = m[4] = 1.05914×10^{12}

-----: avg = 0.000191778

Spectral Moments by Differentiation
(the results are exact for this signal length, LL);
This method does consume CPU time! The larger n2, the larger the CPU time!

-----: m0 = 1.4997

-----: m2 = 58 305.9

-----: m4 = 1.05914×10^{12}

```
-----:  cpu[seconds] = t1 - t0 = 27
```

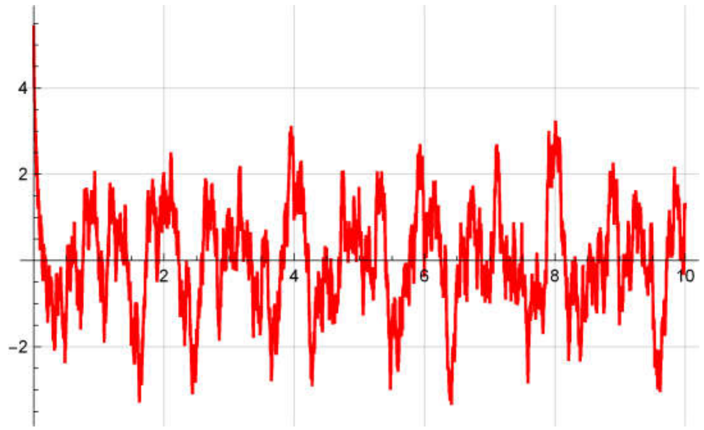
Relative difference between differentiation method and exact solution (the larger LL, the smaller the difference):

reldiff0=Abs[sm0-m0]/sm0 = 0.000475682 .....

reldiff2=Abs[sm2-m2]/sm2 = 9.27914×10^{-6} .....

reldiff4=Abs[sm4-m4]/sm4 = 1.47918×10^{-6} .....

-----: n2 = 26



The EXACT Spectral Moments according to the IG paper (it is implied that LL->Infinity);

These are entirely analytical, so there is NO CPU time consumed, at all!

-----: sm0 = m[0] = 1.49997

-----: sm2 = m[2] = 2.24293×10^{6}

-----: sm4 = m[4] = 6.01752×10^{16}

-----: avg = 0.00019178

Spectral Moments by Differentiation
(the results are exact for this signal length, LL);
This method does consume CPU time! The
larger n2, the larger the CPU time!

-----: m0 = 1.50069

-----: m2 = 2.24293×10^{6}

-----: m4 = 6.01752×10^{16}

-----: cpu[seconds] = t1 - t0 = 56

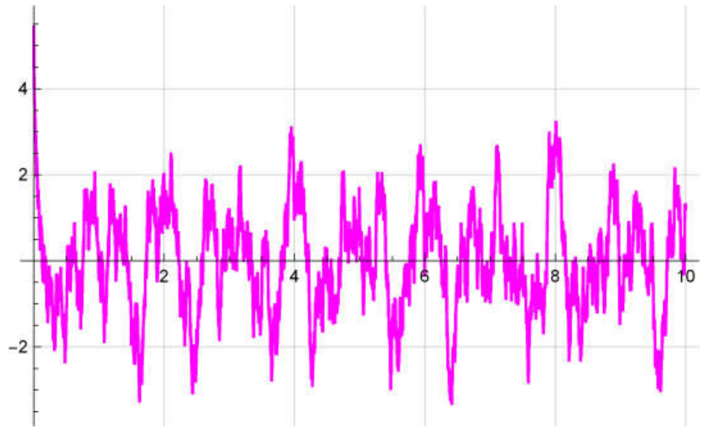
Relative difference between differentiation method and exact solution (the larger LL, the smaller the difference):

reldiff0=Abs[sm0-m0]/sm0 = 0.000475374 .....

reldiff2=Abs[sm2-m2]/sm2 = 2.14607×10^{-7} .....

reldiff4=Abs[sm4-m4]/sm4 = 5.12095×10^{-8} .....

-----: n2 = 34



The EXACT Spectral Moments according
to the IG paper (it is implied that LL->Infinity);
These are entirely analytical, so there is
NO CPU time consumed, at all!

-----: sm0 = m[0] = 1.5

-----: sm2 = m[2] = 5.74849×10^{7}

-----: sm4 = m[4] = 1.013×10^{21}

-----: avg = 0.00019178

Spectral Moments by Differentiation
(the results are exact for this signal length, LL);
This method does consume CPU time! The larger n2, the larger the CPU time!

-----: m0 = 1.50071

-----: m2 = 5.74849×10^{7}

-----: m4 = 1.013×10^{21}

-----: cpu[seconds] = t1 - t0 = 89

Relative difference between differentiation method and exact solution (the larger LL, the smaller the difference):

reldiff0=Abs[sm0-m0]/sm0 = 0.000475365 .....

reldiff2=Abs[sm2-m2]/sm2 = 8.47025×10^{-9} .....

reldiff4=Abs[sm4-m4]/sm4 = 5.39462×10^{-9} .....

Fractal plots of all cases above