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On the kinematics and kinetics of mechanical seals, rotors, and wobbling bodies

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Abstract

Mechanical seals, rotors, and wobbling bodies whirl about a point and are characterized by a kinematical constraint that prevents them from having integral motion with respect to the axis of whirl. A valid kinematical model is a prerequisite to subsequent dynamic analyses. Three previous works have suggested distinctly different kinematical models for the same problem. The analysis herein presents yet another kinematical model that preserves (actually enforces) the proper kinematical constraint. Interestingly, it is found that although no integral rotation is allowed about the axis of whirl, the wobbling body possesses a sustained nonzero angular velocity about that axis. The derivation is done for any finite nutation angle and only final results are being degenerated to small tilt angles. The outcome reaffirms the results of a previous work. For this time-invariant problem the notion of virtual velocity and virtual power emerges, and the equations of motion are derived using Lagrange's equations to complement results obtained previously by Newton–Euler mechanics.

Keywords: Kinematics; Kinetics; Wobbling bodies; Rotors; Whirl; Seals

1. Introduction

The vast majority of mechanical seals have typically at least one element flexibly mounted; it is either the stator or the rotor. Fig. 1a shows a schematic of a flexibly mounted stator (FMS) seal where the ring (stator) is supported by springs and a secondary seal, both of which provide the necessary flexibility for the ring to track a misaligned seat (rotor). Depending upon the application (specifically in high temperature environments) metal bellows are used instead of elastomeric secondary seals. In seal applications it is necessary to prevent the ring from being dragged about the axis of rotation. Bellows accomplish this inherently, because the bellows structure is sufficiently stiff in the circumferential direction, but it is very flexible in the axial direction. The same anti-rotation can be accomplished also by the elastomeric seals (e.g., O-rings), provided that the squeeze in the secondary seal is sufficiently high to prevent ring drag. Metal bellows or O-ring seals are axisymmetric

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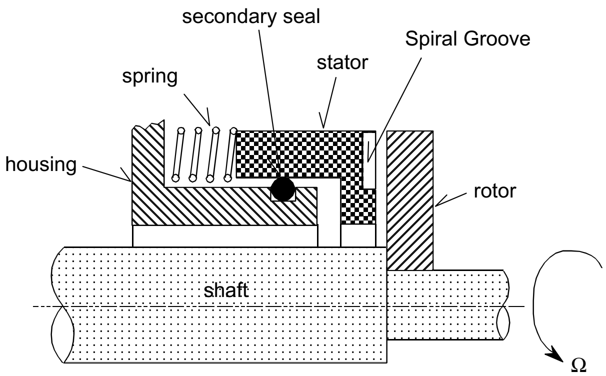
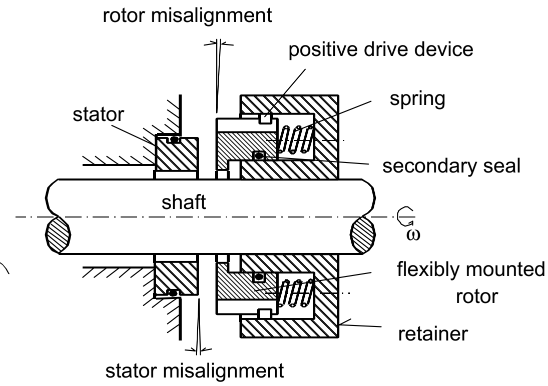


(a) Flexibly mounted stator configuration



(b) Flexibly mounted rotor configuration

Fig. 1. Schematic of mechanical face seals.

structures and are, therefore, indifferent about which transverse axis the stator may tilt. Frequently anti-rotation pins or positive drive devices are added to ensure that the flexibly mounted element is not being dragged around relative to the housing. Such pins protrude from the housing (or retainer) into machined slots in the ring as shown in Fig. 1b that depicts a flexibly mounted rotor (FMR) seal. Here the rim or frame that houses the rotor rotates with the shaft while the stator is fixed. In which case, the pins act as positive drive devices. From a kinematical point of view the FMS configuration is a special (degenerate) case of the FMR case because by letting the rim speed in the latter go to zero, one obtains the desired results for the former. In either case the flexibly mounted element may have only limited motion where a complete (integral) rotation relative to the frame is prevented by the anti-rotation mechanism. This kinematical constraint must, therefore, be reckoned with in the derivations of the equations of motion.

The equation of kinematical constraint is defined by the nature of the anti-rotation device. Flywheels, impellers, magnetic disks on spindles, pin-on-disk testers, etc., can all be described fundamentally as rotating discs that are typically attached to shafts by some mechanism that allows them to tilt about an axis perpendicular to the shaft, and whirl about the center point. Attachment mechanisms may be keyways, press fits, bolts, welding, etc. Once again, although these mechanisms allow some flexibility in a tilt mode they do not allow the disc to complete an integral revolution about that point relative to the shaft. Had there been an integral (complete) rotation, regardless of the application, the anti-rotation devices (pins, keys, etc.) would shear off, a condition which is not permitted in a functional system (and in the current analysis). All machine elements that obey such a constraint are classified as wobbling bodies.

Generally the "point" about which tilt takes place and motion is transmitted, in a basic sense, represents a joint (e.g., Cardan suspension, Hooke joint, or a constant velocity joints). However, the locking devices mentioned above are only a few out of the numerous mechanisms that fulfill the same or similar function. As indicated for seals, instead of pins there can be bellows, piston rings, or O-rings that support the flexibly mounted element. Likewise, other machine elements, e.g., welding, bolts, or press fits are frequently being used to attach rotors to rotating shafts. Since anti-rotation pins allow free rotation (tilt motion) only in a prescribed order, they are not axisymmetric joints. On the other hand bellows, O-rings, welds, and press fits do not impose any preferential order on rotations and can be considered axisymmetric (isotropic) joints. Clearly it is impossible to account for all mechanisms in a single kinematical model without compromise. If, however, one accepts the fact that tilts are limited to small angles, a single kinematical model is possible. There is one constraint, though, that cannot be violated. This is the fact that as long as the locking devices are functional, no shearing can take place between the wobbling body and the housing or shaft upon which the body is attached. This leads to a transmission law that must be satisfied. Note that this work is limited to rigid bodies, i.e., flutter that may be caused by elastic modes is not considered. Also, even though "seal" terminology is used and reference is made to a "seal ring," or a "flexibly mounted rotor," the analysis is valid for all wobbling bodies which have whirl motion including flywheels, impellers, tilted rotors, etc.

2. Kinematical models

Fig. 2 presents a kinematical model for a flexibly mounted element that is constrained by two anti-rotation pins. Its angular position is described by the three Euler angles, i.e., precession, ψ; nutation, γ and spin, ϕ. Moments are sought in a rotating coordinate system, xyz, which precesses relative to coordinate system XYZ that is attached to the outer rim (or housing).

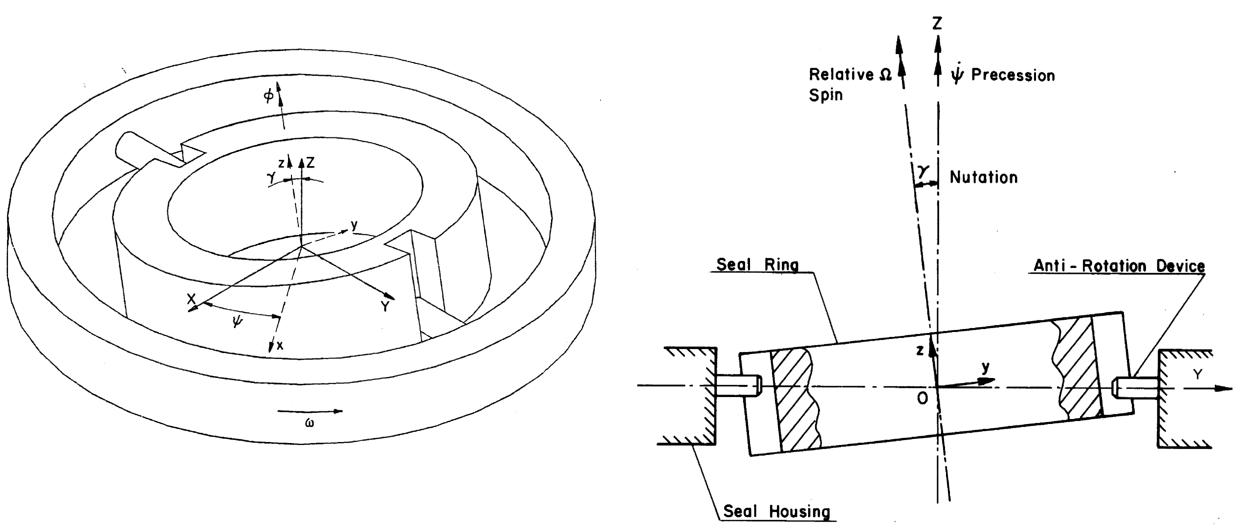
System XYZ (attached to the housing) is either rotating with the shaft, having a velocity $\omega = \text{const}$ about axis Z (FMR), or as a special degenerate case, $\omega = 0$ (FMS). System xyz is whirling (wobbling) within XYZ such that x is the axis about which the nutation (tilt) occurs, z is the out-normal axis about which the relative spin $\Omega = \dot{\phi}$ takes place, and y points toward the point of maximum separation between planes xy and XY.

Had the rotor been a free rotating body, e.g., a sleeping top, then the three Euler angles $\{\gamma, \psi, \phi\}$ would have been kinematically independent. The case at hand, however, is not such. The anti-rotation devices impose a rotational limitation on the rotor and, hence, the Euler angles are not all independent, i.e., they are related through an equation of kinematical constraint. A special but common case is that of precession at steady nutation, i.e., $\gamma = \text{const}$. In which case, after a complete cycle of wobble has completed, each point in the rotor returns precisely to its initial state. This renders a relationship between the spin and the precession, given in a functional form as $\phi = \phi(\psi, \gamma)$, where γ is a parameter. Since the final objective is to obtain the equations of motion, it is necessary to obtain the relationship between $\dot{\phi}$ and $\dot{\psi}$, the ratio of which, $T_r = -\dot{\phi}/\dot{\psi}$, is defined as the transmission law. Clearly the kinematical constraint, or transmission law, must hold for any nutation value, γ, whether it is finite or small. Hence, the analysis herein is done first for any angle γ, and only the final results are degenerated to "small angles".

Three different works have proposed kinematical constraints for mechanical seals that are fundamentally different. The earliest kinematical models with transmission laws are given by Green and Etsion [1]. For a case where the suspension is perfectly axisymmetric and tilt can take place about any axis orthogonal to the whirl or precession axis, for example, bellows, O-rings, press fit, the constant velocity joint provides an ideal (isotropic) transmission law,

$$T_r = -\dot{\phi}/\dot{\psi} = 1. \tag{1a}$$

In the case where the order of tilts is important, such as in anti-rotation or positive drive pins, a Hooke joint provides a suitable non-isotropic model. For which case Ref. [1] derives for small tilts,



(a) 3D view (b) 2D cross-section

Fig. 2. A kinematical model for a seal wobbling element.

$$T_r = -\dot{\phi}/\dot{\psi} = 1 + \left(\sin^2\psi - \frac{1}{2}\right)\gamma^2 - \frac{\gamma\dot{\gamma}}{\dot{\psi}}\sin\psi\cos\psi. \tag{1b}$$

Later work by Lipschitz [2], proposes a transmission law given by

$$T_r = -\dot{\phi}/\dot{\psi} = 1/\cos\gamma, \tag{2}$$

where recently Xiong and Salant [3] propose yet another transmission law, given by

$$T_r = -\dot{\phi}/\dot{\psi} = \cos\gamma. \tag{3}$$

Note that since the kinematical constraint under consideration is between XYZ and xyz, it renders the issue of whether XYZ is inertial or rotating as immaterial. The above transmission laws are identical only for the trivial case when $\gamma = 0$, but are distinctly different otherwise.

It is useful to introduce yet another mechanism which guarantees (in fact, enforces) that any point in the body shall return to its original position after completing one cycle of wobble. This model consists of two identical cones, as shown in Fig. 3. By definition, the space cone is stationary where the body cone wobbles in pure rolling motion about the space cone. The two cones can be considered to consist of two identical bevel gears, where the curved arm generates the precession (input), while the body cone as it whirls around, spins at a rate $\dot{\phi}$ about its own axis z (output). The absolute angular velocity, $\vec{\lambda}$, acts along the instantaneous axis of rotation, which coincides with the cones common generator. Since the cones are identical and motion is constrained to pure rolling, after a half cycle, $\psi = \pi$, points B and B′ shall coincide, and after a complete cycle, $\psi = 2\pi$, point A (on the body cone) shall return to the same location as it is shown in Fig. 3. Note also that under pure rolling all points along the instantaneous axis of rotation have zero velocity, where specifically $\vec{v}_A = 0$.

The problem can be approached from the perspective of an observer that is attached to the rotating system xyz (i.e., the curved arm). Relative to himself the observer sees the space cone as having a precession, $|\dot{\psi}|$,

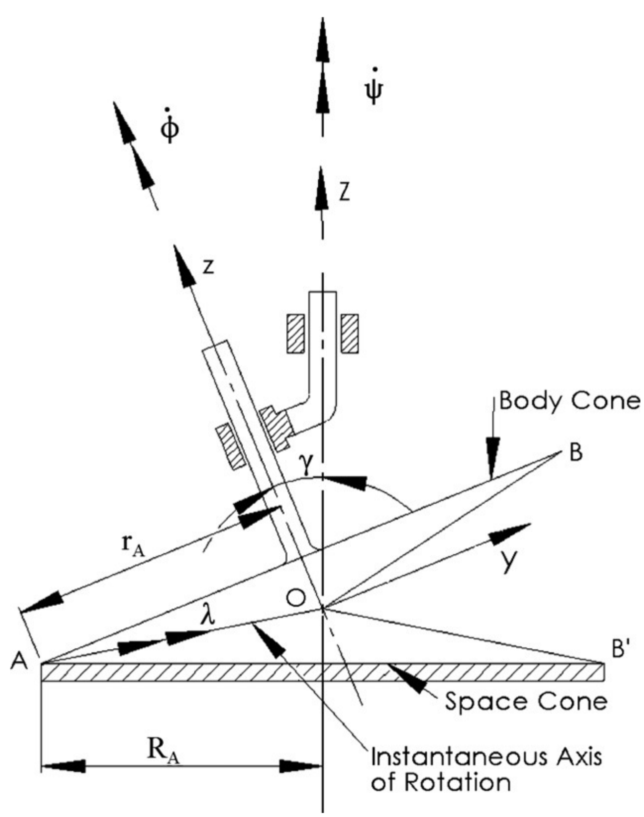


Fig. 3. Identical space and body cones kinematical model.

about negative $\widehat{Z}$, the body cone spins at $\dot{\phi}$ about $\hat{z}$ (direction is yet to be determined), and point A has a velocity relative to xyz directed into the page (i.e., negative $\hat{x}$). In a mathematical form

$$v_A|_{rel} = r_A\dot{\phi} = R_A(-\dot{\psi}).$$

Since geometrically $R_A = r_A$, then $\vec{\dot{\phi}} = -\dot{\psi}\hat{z}$, and thus

$$T_r = -\dot{\phi}/\dot{\psi} = R_A/r_A = 1.$$

Evidently, this transmission law is identical to that derived in Ref. [1] and given in Eq. (1a), and thus the mechanism in Fig. 3 has effectively characteristics of a constant velocity joint. The analysis herein renders an unwavering result: for a wobbling body, which does not have an integral rotation about the axis of wobble (axis Z), the geometrical condition required is $r_A = R_A$. It should be noted that from a kinematical point of view the cones are in fact virtual, where the wobbling body center of mass is positioned at point O (the apex). Superimposing Fig. 2b upon Fig. 3 (by overlaying points O, and axes y and z, respectively) renders the ultimate kinematical model for the body wobbling in conical motion.

The angular velocity of the coordinate system xyz (shown in Fig. 3) takes place about axis Z, and it is given by

$$\vec{\omega}_{xyz} = \dot{\psi}\widehat{Z} = \dot{\psi}(\sin\gamma\hat{y} + \cos\gamma\hat{z}).$$

The angular velocity of the body cone is directed along the instantaneous axis of rotation, defined by the common generator of the two cones. This velocity is given by $\vec{\lambda} = \vec{\omega}_{xyz} + \vec{\Omega} = \vec{\omega}_{xyz} + \dot{\phi}\hat{z}$. Hence, with $\vec{\dot{\phi}} = -\dot{\psi}\hat{z}$ (see above) we have,

$$\vec{\lambda} = \dot{\psi}\sin\gamma\hat{y} + (\dot{\psi}\cos\gamma + \dot{\phi})\hat{z} = \dot{\psi}[\sin\gamma\hat{y} + (\cos\gamma - 1)\hat{z}].$$

Note that the angle between $\vec{\lambda}$ and $\hat{y}$ can easily be verified to be $\cos^{-1}[\hat{y}\cdot\vec{\lambda}/|\hat{y}||\vec{\lambda}|] \equiv \gamma/2$, and more importantly, the projection of $\vec{\lambda}$ upon $\widehat{Z}$ is given by

$$\lambda_Z = \vec{\lambda}\cdot\widehat{Z} = \dot{\psi}[\sin\gamma\hat{y} + (\cos\gamma - 1)\hat{z}]\cdot(\sin\gamma\hat{y} + \cos\gamma\hat{z}) = \dot{\psi}(1 - \cos\gamma).$$

Clearly this component does not vanish for any non-trivial angle $\gamma \neq 0$ (large or small). Hence, it is concluded that even though integral rotation of a wobbling body is not permitted about axis Z, the body's angular velocity component about this axis, λ_Z, has a sustainable non-vanishing value.

The kinematical model presented above using a fixed angle γ is done for convenience in illustration and for establishing the concept. It is clear that T_r for an isotropic joint is independent of the tilt (nutation), γ, i.e., it is valid for any arbitrary angle, γ, as long as the two virtual cones remain identical. Therefore, the above transmission law holds true also when the angle γ varies in time. Hence, for every point in a wobbling body to return to its original position after completing one cycle of whirl (abiding by the anti-rotation devices, gear teeth, etc.) the joint imposes

$$\frac{1}{2\pi}\int_0^{2\pi} T_r \mathrm{d}\psi = 1. \tag{4}$$

Simply by inspection it is evident that Eq. (1a) fulfills this condition for an isotropic joint. Just as well, Eq. (4) is satisfied also for a non-isotropic (Hooke) joint given by Eq. (1b) that does depend upon the nutation angle. The transmission laws given by Eqs. (2) and (3), however, do not obey Eq. (4), and thus are not applicable to wobbling bodies under the said constraint. This conclusion is further reinforced in the Appendix by additional mathematical and physical details. The analysis that follows uses the axisymmetric constraint of Eq. (1a); however, when simplification is done for small angles, the final results are just as well applicable to a kinematical constraint given also by Eq. (1b) because of the aforementioned line of reasoning.

3. Flexibly mounted rotor

It should be noted that the kinematical model above is relative to the rotating housing or the retainer (see Fig. 2a) which rotates with the shaft at ω. This rotation is now being added to the model. The angular velocity of the coordinate system xyz, is

$$\vec{\omega}_{xyz} = \dot{\gamma}\hat{x} + \dot{\psi}_r \sin\gamma\hat{y} + \dot{\psi}_r \cos\gamma\hat{z},$$

where the nutation angle is being relaxed to assume time dependency, and $\dot{\psi}_r = \dot{\psi} + \omega$ is the absolute precession rate about axis $\widehat{Z}$. Since scalarly $\dot{\phi} = -\dot{\psi}$ then $\dot{\phi} = \omega - \dot{\psi}_r$. As previously, adding the spin to $\vec{\omega}_{xyz}$ gives the angular velocity of the rotor,

$$\vec{\lambda} = \vec{\omega}_{xyz} + \dot{\phi}\hat{z} = \dot{\gamma}\hat{x} + \dot{\psi}_r \sin\gamma\hat{y} + [\dot{\psi}_r(\cos\gamma - 1) + \omega]\hat{z}. \tag{5}$$

This, again, inherently contains the effect of the kinematical constraint. It is apparent that from the FMR case, Eq. (5) degenerates to the FMS case by setting $\omega = 0$ (see previous result above).

4. Lagrange's equations

The work by Green and Etsion [1] also provides the dynamic moments about xyz using Newton–Euler mechanics. Ref. [4] attempts the same but it uses instead the Lagrange equations; however, the outcome is enigmatic since the aforementioned kinematical constraint is unaccounted for. A classical way to include kinematical constraints would be through Lagrange multipliers. In the following derivation, however, a direct approach is taken where the Lagrange equations shall account for the kinematical constraint because, as noted, the angular velocity given by Eq. (5) inherently contains the said constraint. System xyz (Fig. 2) is principal for the wobbling body, possessing correspondingly inertia values $\{I, I, I_z\}$, where I and I_z are the transverse and polar moment of inertia, respectively. Using the angular momentum $\{h\} = [I]\{\lambda\}$, and noting that the center of mass, which is coincident with the origin point O, is stationary, leads to the kinetic energy,

$$T = \frac{1}{2}\vec{h}\cdot\vec{\lambda} = \frac{1}{2}\left\{I\dot{\gamma}^2 + I\dot{\psi}_r^2\sin^2\gamma + I_z\left[\dot{\psi}_r(\cos\gamma - 1) + \omega\right]^2\right\}. \tag{6}$$

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \tag{7}$$

where index $j = 1, 2, 3$, signifies respectively the generalized coordinates $\{q_1, q_2, q_3\} \triangleq \{\gamma, \psi_r, \theta\}$, and $\dot{\theta} = \omega$. Similarly, let the index $i = 1, 2, 3$, signify respectively the axes x, y, z, about which the moments are desired. It is obvious that the moment in the tilt direction $M_x = Q_\gamma$ because by definition the generalized coordinate, γ, takes place about (i.e., its direction is along) axis x. However, M_y is not along any of the generalized coordinates, q_j, and thus it requires some special treatment. To obtain the generalized forces, a consistent formulation is used (see Ginsberg [5], pp. 257–269),

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}. \tag{8a}$$

Here, however, the constraint is readily available in its time rate of change, $\dot{\vec{r}} \triangleq \vec{\lambda}$, rather than by its integral form, r_i. Hence, further derivation is necessary. Consider general (rheonomic) constraints of the form $r_i = r_i(q_j, t)$, where $q_j = q_j(t)$, and t is a parameter (namely time). The time rate of change of $r_i = r_i(q_j, t)$ is

$$\dot{r}_i \triangleq \frac{\mathrm{d}r_i(q_j, t)}{\mathrm{d}t} = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}.$$

Taking the derivative of the above with respect to $\dot{q}_j$ (and noting that $r_i = r_i(q_j, t)$ is not a function of $\dot{q}_j$) gives

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left[\sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right] = \frac{\partial r_i}{\partial q_j}.$$

Substituting the above in Eq. (8a) results in

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i F_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \sum_i M_i \cdot \frac{\partial \lambda_i}{\partial \dot{q}_j}. \tag{8b}$$

If commonly the first term in the generalized force is derived from principles of ''virtual displacement'' and ''virtual work,'' then effectively the second term may be interpreted as if it had been derived from principles of ''virtual velocity'' and ''virtual power.'' Substituting now Eq. (5) in (8b), and cycling $j = 1, 2, 3$, results in

$$\begin{aligned} Q_\gamma &= M_x, \\ Q_{\psi_r} &= M_y \sin\gamma + M_z(\cos\gamma - 1), \\ Q_\theta &= M_z. \end{aligned} \tag{9}$$

Applying Eq. (7) to (6), and with the aid of (9), the equations of motion expressed in system xyz are obtained for finite angles,

$$\begin{aligned} & I\ddot{\gamma} - I\dot{\psi}_r^2 \sin\gamma\cos\gamma + I_z[\dot{\psi}_r(\cos\gamma - 1) + \omega]\dot{\psi}_r \sin\gamma = M_x, \\ & I\ddot{\psi}_r \sin^2\gamma + 2I\dot{\psi}_r\dot{\gamma}\sin\gamma\cos\gamma + I_z\{[\ddot{\psi}_r(\cos\gamma - 1) - \dot{\psi}_r\dot{\gamma}\sin\gamma + \dot{\omega}](\cos\gamma - 1) \\ & \quad - [\dot{\psi}_r(\cos\gamma - 1) + \omega]\dot{\gamma}\sin\gamma\} = M_y \sin\gamma + M_z(\cos\gamma - 1), \\ & I_z[\ddot{\psi}_r(\cos\gamma - 1) - \dot{\psi}_r\dot{\gamma}\sin\gamma] = M_z. \end{aligned} \tag{10}$$

Simplifying these equations for small angles, where $\sin\gamma = \gamma + O(\gamma^3)$, and $\cos\gamma = 1 - \gamma^2/2 + O(\gamma^4)$, yields ultimately,

$$\begin{aligned} M_x &= I(\ddot{\gamma} - \dot{\psi}_r^2\gamma) + I_z\omega\dot{\psi}_r\gamma + O(\gamma^2), \\ M_y &= I(\ddot{\psi}_r\gamma + 2\dot{\psi}_r\dot{\gamma}) - I_z\omega\dot{\gamma} + O(\gamma^2), \\ M_z &= -I_z(\dot{\psi}_r\dot{\gamma}\gamma + \ddot{\psi}_r\gamma^2/2) + O(\gamma^2) \approx O(\gamma^2). \end{aligned} \tag{11}$$

These equations match identically those in Ref. [1] [see there Eq. (24)], which had been obtained by Newton–Euler mechanics. To degenerate the results further to FMS, substitute $\omega = 0$, which nullifies the gyroscopic terms in Eqs. (11) (and by definition also $\dot{\psi}_r \equiv \dot{\psi}$). It is worthy to note that Eq. (11) above have been furthermore transformed into an inertial frame [6], and have subsequently provided the foundation for deriving the equations of motion in shaft fixed system for a whirling (overhanging) rotor [7].

5. Conclusions

A valid kinematical model that represents the physical constraint imposed by the anti-rotation or locking devices is fundamental to the proper derivation of the equations of motion. A new kinematical model is proposed which consists of two identical cones (a body cone and a space cone). With the constraint imbedded in the angular velocity of the wobbling body the use of Lagrange's equations affirms a previous result that had been obtained by Newton–Euler mechanics. It is found that even though a wobbling body (body cone) has no finite (or integral) rotation about the vertical axis Z, it does have a steady nonzero angular velocity about that axis.

Acknowledgement

The artwork prepared by Mr. Yoav Green is gratefully acknowledged.

Appendix. Kinematical models for Refs. [2,3]

A kinematical model is now sought (reversed-engineered) for the transmission laws introduced in Ref. [2]. We shall investigate the mechanism (kinematical model) shown in Fig. 4, where the body cone rolls upon the semi-infinite plane, i.e., the space cone.

The rotating coordinate system, xyz, is consistent with the definitions herein, i.e., xyz is attached to the curved arm as it precesses about axis Z relative to the inertial coordinate system XYZ. Note that the body cone is free to spin within xyz about axis z. The angular velocity of the coordinate system xyz, which is entirely about axis Z, is conveniently expressed also in xyz, giving

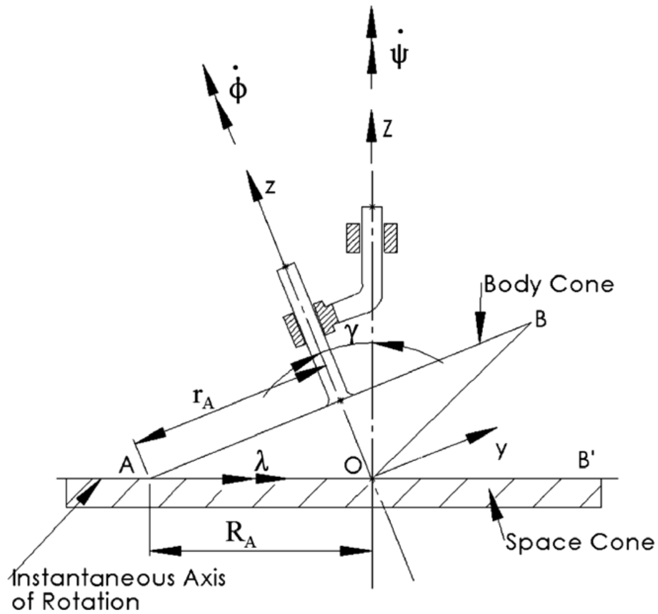


Fig. 4. Cone on plane kinematical model.

$$\vec{\omega} = \dot{\psi}\widehat{Z} = \dot{\psi}(\sin\gamma\hat{y} + \cos\gamma\hat{z}). \tag{A1}$$

The angular velocity of the body cone is directed along the instantaneous axis of rotation, defined by the common generator of the two cones. This velocity is given by

$$\vec{\lambda} = \dot{\psi}\sin\gamma\hat{y} + (\dot{\psi}\cos\gamma + \dot{\phi})\hat{z}, \tag{A2}$$

where $\dot{\phi}$ is the (yet to be determined) spin of the body cone about axis z. In this mechanism, since $\vec{\omega}$ and $\vec{\lambda}$ are orthogonal to each other, then $\vec{\omega}\cdot\vec{\lambda} = 0$. This straightforwardly leads to the transmission law

$$T_r = -\dot{\phi}/\dot{\psi} = 1/\cos\gamma \tag{A3}$$

being identical to the one given by Eq. (2) and in Ref. [2]. Hence, it is proven that the mechanism in Fig. 4 provides a true representation of the transmission law proposed by Ref. [2].

Now it is useful to investigate that mechanism by a different approach. It is noted that under pure rolling all points along the instantaneous axis of rotation have zero velocity, where specifically $\vec{v}_A = 0$. An observer placed upon the rotating system xyz sees (relative to himself) the following: the space cone has a precession, $-\dot{\psi}$, about the vertical axis Z; the body cone has a spin $\dot{\phi}$ about axis z; and that point A has a velocity relative to xyz,

$$v_A|_{rel} = r_A\dot{\phi} = R_A(-\dot{\psi}).$$

Since geometrically $R_A/r_A = 1/\cos\gamma$, then with the aid of Eq. (A3), the apparent outcome is that $T_r = R_A/r_A$. This is of utmost significance, since for this mechanism $r_A \neq R_A$. Hence, a point on the body cone, which at some instant is positioned on the space cone at point A, shall not return to that point after the body cone has completed one cycle of wobble because of the unequal circumferences ($2\pi r_A \neq 2\pi R_A$). Likewise, point B on the body cone shall not contact the line AB′ after half cycle, $\psi = \pi$, as point B advances on the semi-infinite plane. This obviously violates the kinematical constraint imposed by anti-rotation devices. Noteworthy, throughout this work, the analysis herein is not restricted to small tilt (nutation) angles, demanding that the kinematical model be valid for any finite angle, γ. Thus the kinematical condition given by Ref. [2], or Eq. (2), cannot represent the problem at hand.

Similarly, it would not be difficult to construct a mechanism that provides $R_A/r_A = \cos\gamma$, as implied by the transmission law of Ref. [3] and given by Eq. (3). For conciseness it is omitted because again $2\pi r_A \neq 2\pi R_A$

and by the aforementioned line of reasoning it is concluded that also Eq. (3) would likewise violate the kinematical constraint.

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