The Rotor Dynamic Coefficients of Eccentric Mechanical Face Seals

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The Reynolds equation is extended to include the effects of radial deflection in a seal with two flexibly mounted rotors. The resulting pressures are used to obtain the forces and moments introduced in the axial and angular modes by the inclusion of eccentricity in the analysis. The rotor dynamic coefficients relating the forces and moments in these modes to the axial and angular deflection are shown to be the same as those presented in the literature for the concentric case. Additional coefficients are obtained to express the dependence of these forces and moments upon the radial deflections and velocities. The axial force is shown to be decoupled from both the angular and radial modes, but the angular and radial modes are coupled to one another by the dependence of the tilting moments upon the radial deflections. The shear stresses acting upon the element faces are derived and used to obtain the radial forces acting upon the rotors. These forces are used to obtain rotor dynamic coefficients for the two radial degrees of freedom of each rotor. The additional rotor dynamic coefficients can be used to obtain the additional equations of motion necessary to include the radial degrees of freedom in the dynamic analysis. These coefficients introduce additional coupling between the angular and radial degrees of freedom, but the axial degrees of freedom remain decoupled.

Introduction

In high-speed applications, the dynamic behavior of a mechanical face seal is an important consideration in predicting its performance. Instability in the seal may lead to premature failure resulting from face contact, and excessive vibration at steady-state will increase both the wear and the leakage of the seal.

Analyses of the dynamic behavior of mechanical face seals date back almost three decades. Extensive reviews of the literature in this field have been provided by Allaire (1984), Tournerie and Frene (1985), and Etsion (1982, 1985, and 1991). More recent work in the field has been done by Green (1987, 1989, and 1990) and by Wileman and Green (1991).

The majority of the literature deals with seals in which the seal ring and the seal seat remain concentric with respect to each other and the shaft. Such a concentric analysis may provide a good prediction of the dynamic behavior of a system with a very stiff structure. Trends in modern turbomachinery, however, are toward increasingly lighter machine components which are likely to be quite flexible. In these systems it is necessary to include the effect of eccentricity upon the seal, as the ring, seat, and shafts are likely to deflect in the radial deflection so that their axes of rotation are no longer coincident.

The literature dealing with analysis of eccentric seals is limited. Findlay (1969) showed that a seal which is both eccentric and misaligned will exhibit a "pumping" action, and he verified this result experimentally. Sneck (1969) verified the pumping effect and showed that eccentricity will also affect the separating force in the seal. He noted, however, that these effects occur only in conjunction with misalignment or waviness. Griskin (1987) determined the dynamic response using numerical integration for a seal design having a fixed eccentricity.

To determine the dynamic response of a lubricated system such as a face seal, it is normally necessary to solve the equations of motion simultaneously with the Reynolds equation, which governs the behavior of the sealed fluid. Such an analysis requires an iterative numerical solution such as that presented by Green and Etsion (1986) for a concentric seal. Green and Etsion (1983) demonstrated that an analytical solution can be obtained in closed form if the fluid behavior is modeled using linearized rotor dynamic coefficients. Their original result was for a concentric seal with a flexibly mounted stator (FMS) and was used to obtain the stability criteria and steady-state response for the system (Green and Etsion, 1985). Green (1987, 1989, and 1990) extended this work to the flexibly mounted rotor (FMR) configuration, finding the FMR configuration superior to the FMS in every aspect of dynamic performance.

Wileman and Green (1991) obtained rotor dynamic coefficients for a seal configuration (denoted FMRR) in which both elements are flexibly mounted to rotating shafts (Fig. 1), but they maintained the assumption of concentricity in the system. These coefficients were used by Wileman (1994) to obtain the equations of motion for the system.

Frequently the designer has the choice of which configuration (FMS, FMR, FMRR) he prefers. If speeds are high enough that dynamic effects are expected to be a problem, a dynamic analysis can assist in this decision. Sometimes, however, the application dictates the seal configuration. In drilling equipment or in gas turbines (Miner, 1992), for example, there is sometimes a need to seal between two rotors, in which case the FMRR configuration is required.

The dynamic analysis of an eccentric system requires two significant extensions of the concentric results. First, the radial deflection of the elements may create additional axial forces and tilting moments which affect the response in the axial and angular modes. Thus, additional rotor dynamic coefficients will be necessary to incorporate eccentricity effects into the equations of motion for these modes. Because the Reynolds equation for incompressible fluids is linear, the eccentricity effects can be obtained independently and combined with the previously derived concentric results using superposition.

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Contributed by the Tribology Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS and presented at the STLE/ASME Tribology Conference, Orlando, FL, October 8–11, 1995. Manuscript received by the Tribology Division February 28, 1995; revised manuscript received June 28, 1995. Paper No. 95-Trib-18. Associate Technical Editor: I. Etsion

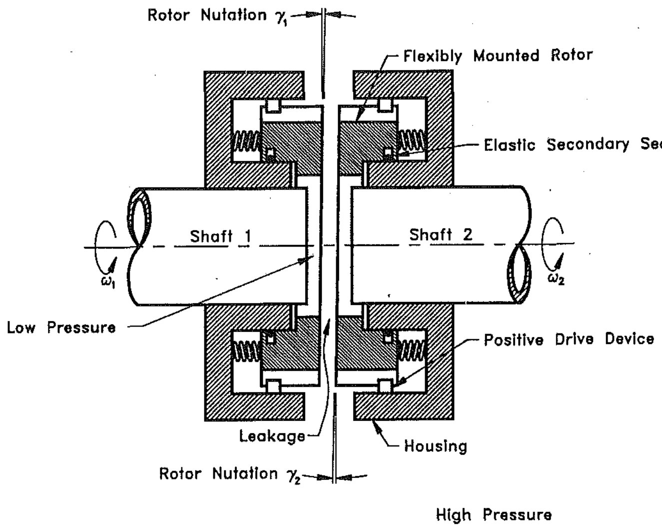

Fig. 1 Schematic of an FMRR mechanical seal

Second, the inclusion of eccentricity adds two degrees of freedom for each rotor. These two degrees of freedom are represented by two perpendicular radial motions, and it is necessary to derive additional rotor dynamic coefficients to model the radial forces and derive the equations of motion in these two directions. This work presents a derivation of both sets of rotor dynamic coefficients.

Kinematic Description

The FMRR configuration is the most general of the kinematic models in the literature; both the FMS and FMR configurations represent degenerate cases of the FMRR configuration. Thus, by using the FMRR configuration as the basis for the eccentric analysis, results can be obtained which will be applicable to all of the possible configurations.

In a concentric analysis it is sufficient to use a single cylindrical coordinate system with inner radius, r_i, and outer radius, r_o. These radial boundaries are based upon the geometry of the sealing dam, which is limited by the width of the seal ring. When the ring and the seat are concentric, it is not necessary to specify which of the elements is the seat and which the ring, since this will not affect the geometry of the sealing dam.

An eccentric analysis, however, requires a way to express position and velocity on the surfaces of two elements which have different centers. The position vector of a point on the sealing dam surface will depend upon which of the centers is used as the origin of the reference frame. Because it is convenient to evaluate integrals over areas which have constant inner and outer radii, we choose a reference frame which has its origin at the center of the seal ring. In such a system, the sealing dam will be axisymmetric about the origin when the faces are parallel. The relative eccentricity vector, $\boldsymbol{\epsilon}$, is defined as the vector from this origin to the center of the seal seat (Fig. 2). For convenience in the derivation, we will establish the convention that element 1 is the seal ring and element 2 is the seat. Once the origin has been chosen, the $(123)_\epsilon$ system is defined in which the $\hat{2}_\epsilon$ axis is parallel to the direction of the relative eccentricity, so that there is no component of the relative deflection in the direction of the perpendicular $\hat{1}_\epsilon$ axis (Fig. 3).

If we define the center of element 1, the seal ring, as O and the center of element 2, the seat, as O', then the vector describing the eccentricity is

$$\mathbf{r}_{OO'} = \epsilon \hat{2}_\epsilon$$

where ϵ is the distance between the two centers. The velocity of O' with respect to O will be

$$\mathbf{v}_{O'/O} = \frac{d\mathbf{r}_{OO'}}{dt} = \frac{\partial \mathbf{r}_{OO'}}{\partial t} + \boldsymbol{\omega}_{(123)_\epsilon} \times \mathbf{r}_{OO'} \qquad (1)$$

since the $(123)_\epsilon$ system will rotate unless the shaft centers are

Nomenclature

C = instantaneous seal centerline clearance
C_0 = equilibrium centerline clearance
d_{ij} = fluid film damping coefficient (element 2)
$d_{\epsilon n}$ = damping coefficient relating tilting moments and eccentric deflections
$d_{\epsilon nr}$ = damping coefficient relating radial forces and eccentric deflections
F_j = generalized force
F_ϵ^* = dimensional axial force
F_ϵ = normalized axial force, F_ϵ^*/Sr_0^2
$F_{1,2}$ = radial forces
G_0 = damping parameter, Eq. (27)
h = film thickness
H = normalized film thickness, h/C
k_{ij} = fluid film stiffness coefficient (element 2)
$k_{\epsilon n}$ = stiffness coefficient relating tilting moments and eccentric deflections
$k_{\epsilon nr}$ = stiffness coefficient relating radial forces and eccentric deflections
$k_{\gamma nr}$ = stiffness coefficient relating radial forces and rotor tilt
M_{nj}^* = dimensional moment
M_{nj} = normalized moment, M_{nj}^*/Sr_0^3
p = pressure
P = normalized pressure, p/S
r = radius
R = normalized radius, r/r_o
S = seal parameter, $6\mu\omega_{\text{ref}}(r_o/C_0)^2(1 - R_i)^2$
t^* = dimensional time
t = normalized time, $\omega_{\text{ref}} t^*$
V_{ni} = translational velocity on seal surface
W_0 = eccentric force parameter, Eq. (34)
z = relative axial translation
Z = normalized axial translations, z/C_0
β^* = coning angle
β = normalized coning angle, $\beta^* r_o/C_0$
γ^* = nutation (tilt)
γ = normalized nutation, $\gamma^* r_o/C_0$
δ = coning parameter, $\beta^* r_o/C$
ϵ^* = dimensional relative eccentric deflection, $\epsilon = \epsilon^*/C_0$
ϵ = normalized relative eccentric deflection
ϵ_n = normalized absolute eccentric deflection
κ = tilt parameter, $\gamma^* r_o/C$
θ = angular coordinate referenced to axis 2
θ_ϵ = angular coordinate referenced to axis 2_ϵ
μ = viscosity
ϕ_n = precession angle
$\psi_{a\epsilon}$ = absolute precession of relative eccentricity system
ω_n = shaft angular speed of element n
ω_{ref} = reference shaft speed (used for normalization)

Subscripts

0 = equilibrium value
i = inner radius
m = mean radius
n = element number (n = 1 or 2)
o = outer radius
ϵ = axis or variable in eccentric analysis

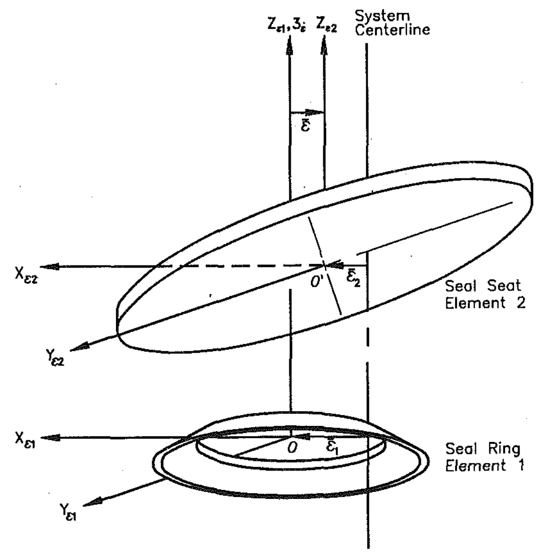

Fig. 2 Eccentric seal kinematic model and coordinate systems

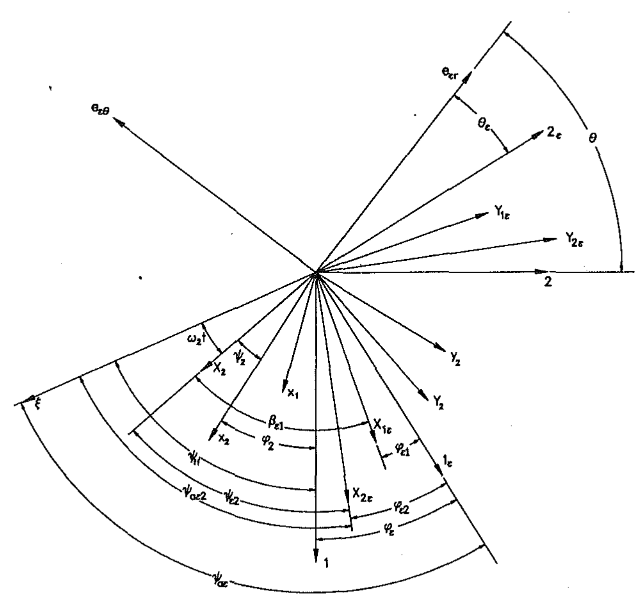

Fig. 4 Vector diagram of the eccentric and shaft-fixed reference frames

fixed in space, in which case the eccentricity vector will be constant. $\psi_{a\epsilon}$ is defined as the angle between this system and the inertial axis ξ as illustrated in Fig. 4 where, for completeness, the vectors and angles for the eccentric analysis have been appended to those described by Wileman and Green (1991) for the concentric analysis.

Since the relative velocity between the two centers in (1) results from eccentricity, we shall adopt the less cumbersome notation $\mathbf{v}_\epsilon$ to represent it. Substituting the expression for the eccentricity vector and the angular velocity then yields

$$\mathbf{v}_\epsilon = \dot{\epsilon}\hat{2}_\epsilon + \dot{\psi}_{a\epsilon}\hat{3} \times \epsilon\hat{2}_\epsilon = -\epsilon\dot{\psi}_{a\epsilon}\hat{1}_\epsilon + \dot{\epsilon}\hat{2}_\epsilon$$

The relative velocity between the two seal elements induces shear stresses which produce radial forces acting upon the faces of the elements. To compute these shear stresses it is necessary to express $\mathbf{v}_\epsilon$ in a cylindrical coordinate system, where r is measured from the origin O and θ_ϵ is measured from axis $\hat{2}_\epsilon$ (Fig. 4). The relationship between the cartesian system and the cylindrical system is

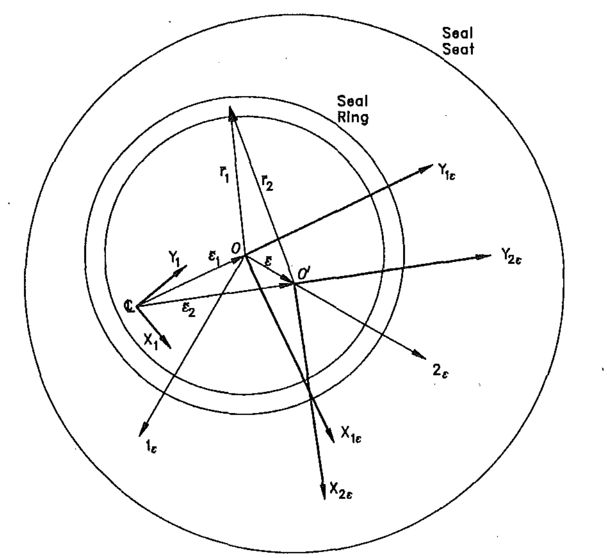

Fig. 3 Vector diagram of relative and absolute eccentric deflections

$$\begin{Bmatrix} \hat{1}_\epsilon \\ \hat{2}_\epsilon \end{Bmatrix} = \begin{pmatrix} -\sin\theta_\epsilon & -\cos\theta_\epsilon \\ \cos\theta_\epsilon & -\sin\theta_\epsilon \end{pmatrix} \begin{Bmatrix} \hat{e}_{\epsilon r} \\ \hat{e}_{\epsilon\theta} \end{Bmatrix} \quad (2)$$

where $\hat{e}_{\epsilon r}$ and $\hat{e}_{\epsilon\theta}$ are unit vectors in the radial and circumferential directions, respectively. When expressed in this system, the relative velocity between the centers of mass is

$$\mathbf{v}_\epsilon = (\epsilon\dot{\psi}_{a\epsilon}\sin\theta_\epsilon + \dot{\epsilon}\cos\theta_\epsilon)\hat{e}_{\epsilon r} + (\epsilon\dot{\psi}_{a\epsilon}\cos\theta_\epsilon - \dot{\epsilon}\sin\theta_\epsilon)\hat{e}_{\epsilon\theta} \quad (3)$$

The relative velocities of adjacent points on the two elements will include this velocity difference between the centers, but will have an additional component which results from the shaft rotations and which we shall denote $\mathbf{v}_\omega$. The additional velocity is obtained for each element as the cross product of the angular velocity of the seal element and the vector from the center of rotation to the point at which the velocity is desired. Recall that the reference frame origin coincides with the center of element 1, the seal ring, so that the position of a point on element 1 is expressed in the cylindrical system as $\mathbf{r}_1 = r\hat{e}_{\epsilon r}$.

The position vector for a point on element 2 will be the same as that of the adjacent point on element 1, noting that r is always measured from the center of element 1. To compute the velocity at a point on element 2, however, we need the vector to the point from the center of rotation of element 2 rather than the vector from the origin at the center of element 1. Thus, we need to subtract the relative eccentricity vector from the position vector (Fig. 3). Transform the eccentricity vector into the cylindrical system using (2) to obtain

$$\boldsymbol{\epsilon} = \epsilon\hat{2}_\epsilon = \epsilon\cos\theta_\epsilon\hat{e}_{\epsilon r} - \epsilon\sin\theta_\epsilon\hat{e}_{\epsilon\theta}$$

Then subtracting this from the position vector yields the vector needed to obtain the velocity on element 2

$$\mathbf{r}_2 = r\hat{e}_{\epsilon r} - \epsilon\hat{2}_\epsilon = (r - \epsilon\cos\theta_\epsilon)\hat{e}_{\epsilon r} + \epsilon\sin\theta_\epsilon\hat{e}_{\epsilon\theta} \quad (4)$$

The velocities are obtained as simple cross products, noting that the spin axis is always parallel to the $\hat{3}$ direction. For element 1 the velocity is

$$\mathbf{v}_{\omega 1} = \omega_1\hat{3} \times r\hat{e}_{\epsilon r} = r\omega_1\hat{e}_{\epsilon\theta} \quad (5)$$

For element 2, the velocity is

$$\mathbf{v}_{\omega 2} = \omega_2 \hat{3} \times (r\hat{e}_{\epsilon r} - \epsilon \hat{2}_\epsilon) = r\omega_2 \hat{e}_{\epsilon\theta} + \epsilon\omega_2 \hat{1}_\epsilon$$
$$= \omega_2 \hat{3} \times [(r - \epsilon \cos \theta_\epsilon)\hat{e}_{\epsilon r} + \epsilon \sin \theta_\epsilon \hat{e}_{\epsilon\theta}]$$
$$= (r\omega_2 - \epsilon\omega_2 \cos \theta_\epsilon)\hat{e}_{\epsilon\theta} - \epsilon\omega_2 \sin \theta_\epsilon \hat{e}_{\epsilon r} \quad (6)$$

The velocities which result from eccentricity lead to two different types of rotor dynamic coefficients, which we shall derive separately. The first type relates the axial forces and tilting moments to the radial displacement and velocity. These coefficients are obtained by solving the Reynolds equation for the incremental pressures which result from eccentricity, then integrating the pressures to determine the resulting axial forces and tilting moments.

The second set of rotor dynamic coefficients relates the kinematic variables to the forces which act in the radial direction. These coefficients are obtained by deriving the shear stresses which result from the radial motion and integrating them over the sealing faces to determine the resulting radial forces.

Reynolds Equation Solution

To evaluate the effect of the eccentric velocity components upon the axial forces and tilting moments, we start with the Reynolds equation and obtain a solution for the fluid film pressure. Because the Reynolds equation for incompressible fluids is linear, the additional pressure resulting from eccentricity can be obtained independently, along with its resulting forces and moments. Adding these forces and moments to those already obtained for the concentric case yields the totals for the system.

The complete form of the Reynolds equation applicable to the FMRR configuration is presented by Wileman (1994). The equation is expressed in terms of velocity components in the cylindrical system. As in the concentric analysis (Wileman and Green, 1991), the Reynolds equation is simplified using the narrow seal approximation, neglecting the effects of curvature and of the circumferential pressure gradient. The fluid film is assumed to have a hydrostatic pressure sufficient to prevent cavitation.

The total eccentric velocity at any point in the sealing dam is the sum of the rotation component, contained in (5) and (6), and the motion of the center of mass, contained in (3). For the Reynolds equation solution we can assume, without loss of generality, that the relative motion of the centers is applied completely to element 2. In this case, the total velocity of a point on element 1 is

$$\mathbf{v}_1 = r\omega_1 \hat{e}_{\epsilon\theta} \quad (7)$$

and the velocity of a point on element 2 is

$$\mathbf{v}_2 = [\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \theta_\epsilon + \dot{\epsilon} \cos \theta_\epsilon]\hat{e}_{\epsilon r} + [r\omega_2 + \epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \cos \theta_\epsilon - \dot{\epsilon} \sin \theta_\epsilon]\hat{e}_{\epsilon\theta} \quad (8)$$

The velocity terms containing $r\omega_1$ and $r\omega_2$ can be omitted in the eccentric analysis because they are accounted for in the concentric analysis (Wileman and Green, 1991).

When (7) and (8) are substituted into the Reynolds equation, several terms vanish completely, and others add out. The Reynolds equation for the eccentric analysis becomes (Wileman, 1994)

$$\frac{\partial}{\partial r}\left(h^3 \frac{\partial p_\epsilon}{\partial r}\right) = -\frac{6\mu}{r}(V_{2\theta} - V_{1\theta})\frac{\partial h}{\partial \theta} - 6\mu(V_{2r} - V_{1r})\frac{\partial h}{\partial r} \quad (9)$$

where the subscripts of the velocity variables contain the element number and the coordinate direction in the cylindrical system.

To solve the equation, a more precise description of the geometry of the seal elements and the fluid film is necessary. Assume that element 1 is a thin ring which satisfies the narrow seal hypothesis. Then element 2 must be assumed to be a disk

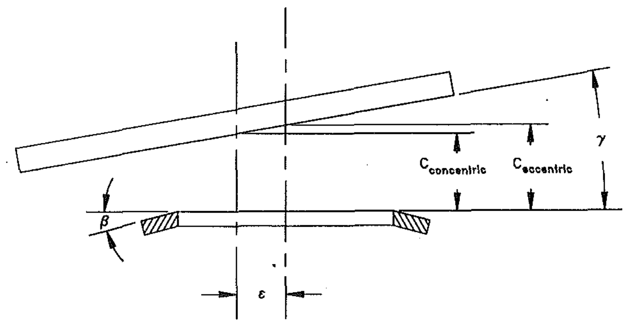

Fig. 5 Effect of eccentricity upon centerline clearance

which extends radially well inside and outside the ring, so that the limiting radial dimensions of the sealing dam are always those of the thin ring. This is a reasonable approximation of a real seal, where the thin ring of element 1 corresponds to the carbon ring of the seal, and the disk of element 2 corresponds to the steel seat.

Using this definition, it is clear that the relative orientation of the two elements does not change when their geometric centers undergo a relative displacement. Wileman and Green (1991) defined the 123 reference frame to describe the relative orientation of the two rotors in the concentric analysis. The relative nutation between the two elements is about axis 1, and axis 2 represents the direction of maximum film thickness. The expression for the film thickness in a concentric seal is

$$h = C + \gamma r \cos \theta + \beta(r - r_i) \quad (10)$$

where the angle θ is measured from axis 2. The second and third terms, which represent tilt and coning effects, respectively, depend only upon the relative orientation of the two elements. Only the first term, which represents the centerline clearance, will change when the shaft centers undergo a relative deflection. If C remains defined as the centerline clearance when the shafts are concentric, the effects of eccentricity can all be combined into a single additional term representing the additional centerline separation which results from eccentricity. The value of this term will depend upon both the magnitude and direction of the eccentric deflections.

The eccentricity is defined in terms of its magnitude, ϵ, and the direction of the deflection, ϕ_ϵ. The angle ϕ_ϵ is measured from axis 2 and represents the relative rotation between the (123) and $(123)_\epsilon$ reference frames (Fig. 4). Then the total centerline clearance is

$$C_{\text{total}} = C_{\text{concentric}} - \gamma\epsilon \cos \phi_\epsilon$$

so that the film thickness becomes (see Fig. 5)

$$h = C_{\text{concentric}} - \gamma\epsilon \cos \phi_\epsilon + \gamma r \cos \theta + \beta(r - r_i)$$

Thus, the only effect of the eccentricity upon the film thickness is a second-order term. Further, the additional term does not change either $\partial h/\partial \theta$ or $\partial h/\partial r$. Since h does not appear directly in the right-hand side of (9), but only as a derivative, the effects of eccentricity upon the definition of the film thickness can be neglected in the Reynolds equation, and the expression for the concentric film thickness, (10), can be used.

Substitute the velocities (7) and (8) and the expression for h into (9). The resulting expression can be simplified further by noting that $\theta_\epsilon = \theta - \phi_\epsilon$ (Figure 4) and substituting the angle difference trigonometric identities. The Reynolds equation becomes

$$\frac{\partial}{\partial r}\left(h^3 \frac{\partial p_\epsilon}{\partial r}\right) = 6\mu\gamma[\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \phi_\epsilon - \dot{\epsilon} \cos \phi_\epsilon] - 6\mu\beta[\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \theta_\epsilon + \dot{\epsilon} \cos \theta_\epsilon] \quad (11)$$

The pressure solution is obtained by integrating the Reynolds equation. Zero boundary conditions are used since the effect of the pressures at the inner and outer radius of the seal has been accounted for in the homogeneous solution of the concentric analysis.

The integration is performed as in Wileman and Green (1991). If the right-hand side of (11) is abbreviated as (R.H.S.), the pressure which results from integrating (11) twice and setting $p_\epsilon = 0$ at $r = r_i$ and $r = r_o$ is

$$p_\epsilon = -(\text{R.H.S.}) \frac{(r_o - r)(r - r_i)}{2h^2 h_m}$$

The pressure is normalized using the definitions

$$P_\epsilon = \frac{p_\epsilon}{S}; \quad R = \frac{r}{r_o}; \quad H = \frac{h}{C_0}; \quad \gamma = \frac{\gamma^* r_o}{C_0};$$

$$\beta = \frac{\beta^* r_o}{C_0}; \quad \epsilon = \frac{\epsilon^*}{C_0} \quad t = \omega_{\text{ref}} t^* \tag{12}$$

where asterisks represent dimensional variables. The seal parameter, S, is defined by

$$S = 6\mu\omega_{\text{ref}} \left(\frac{r_o}{C_0}\right)^2 (1 - R_i)^2$$

and ω_{ref} is a reference speed of the same order of magnitude as the larger of the two shaft speeds. The resulting normalized pressure is

$$P_\epsilon = -\frac{(1 - R)(R - R_i)}{2H_m H^2 (1 + Z)^2 (1 - R_i)^2} \{\gamma[\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \phi_\epsilon - \dot{\epsilon} \cos \phi_\epsilon] - \beta[\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \theta_\epsilon + \dot{\epsilon} \cos \theta_\epsilon]\} \tag{13}$$

The Axial Force and Tilting Moments

The axial force and the moments about the 1 and 2 axes are obtained by integrating the pressure in (13) over the sealing dam surface. For element 2 the definitions are

$$F_\epsilon = R_m \int_0^{2\pi} \int_{R_i}^1 P_\epsilon dR d\theta \tag{14}$$

$$M_{1\epsilon} = R_m^2 \int_0^{2\pi} \int_{R_i}^1 P_\epsilon \cos \theta dR d\theta \tag{15}$$

$$M_{2\epsilon} = R_m^2 \int_0^{2\pi} \int_{R_i}^1 P_\epsilon \sin \theta dR d\theta \tag{16}$$

The numeral in the moment subscript indicates the axis, in the (123) reference frame, about which the moment is applied. The ϵ indicates that the component results from eccentricity effects.

The integration over R in each of these equations can be reduced to the expression $T(\theta)$ defined by (Wileman and Green, 1991)

$$T(\theta) = \int_{R_i}^1 \frac{(1 - R)(R - R_i)}{H_m H^2} dR$$

$$= 2\left[\frac{\ln H_o - \ln H_i}{(\delta + \kappa \cos \theta)^3} - \frac{1 - R_i}{H_m(\delta + \kappa \cos \theta)^2}\right] \tag{17}$$

where $\delta = \beta/(1 + Z)$ and $\kappa = \gamma/(1 + Z)$. Define $\bar{\gamma} = \kappa/\delta$, and assume that $(\kappa/\delta)^2 \ll 1$. Then $T(\theta)$ can be approximated by (Green and Etsion, 1983)

$$T(\theta) = \frac{2\alpha'(1)}{\delta^3} + \frac{2 \cos \theta}{\delta^3} [\alpha(1) - \alpha(R_i) - 3\alpha'(1)\bar{\gamma}] - \frac{2(1 - R_i)}{\delta^2[1 + \delta(R_m - R_i)]}$$

$$+ 2(1 - R_i) \frac{\cos \theta}{\delta^2} \left[\frac{\alpha(R_m) + 2\bar{\gamma}}{1 + \delta(R_m - R_i)}\right] \tag{18}$$

where

$$\alpha(R) = \frac{\kappa R}{1 + \delta(R - R_i)}; \quad \alpha'(R) = \ln [1 + \delta(R - R_i)]$$

This substitution allows the integration over θ to be performed in closed form.

When (17) is substituted into (14), the integral for the axial force becomes

$$F_\epsilon = -\frac{R_m}{2(1 + Z)^2 (1 - R_i)^2} \times \int_0^{2\pi} T(\theta) \{\gamma[\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \phi_\epsilon - \dot{\epsilon} \cos \phi_\epsilon] - \beta[\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \theta_\epsilon + \dot{\epsilon} \cos \theta_\epsilon]\} d\theta \tag{19}$$

Note the integrand in (19) consists of two terms: one resulting from the tilt, γ, and one from the coning, β. The tilt term is independent of θ, and most of this portion of the integrand can be moved outside the integral.

The necessary integrals involving $T(\theta)$ are evaluated by Wileman (1994) and are listed in the appendix. When these are substituted into (19) the tilt term becomes

$$F_{\epsilon\,\text{tilt}} = -\frac{2\pi\gamma R_m}{\delta^3 (1 + Z)^2 (1 - R_i)^2} \left[\alpha'(1) - \frac{\delta(1 - R_i)}{1 + \delta(R_m - R_i)}\right] \times [\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \phi_\epsilon - \dot{\epsilon} \cos \phi_\epsilon] \tag{20}$$

To evaluate the coning term substitute $\theta_\epsilon = \theta - \phi_\epsilon$ and simplify using the angle difference identities. Substituting the appropriate integrals from the appendix yields

$$F_{\epsilon\,\text{coning}} = \frac{\pi\beta R_m}{\delta^3 (1 + Z)^2 (1 - R_i)^2} \times [-\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \phi_\epsilon + \dot{\epsilon} \cos \phi_\epsilon] \times \left\{[\alpha(1) - \alpha(R_i) - 3\alpha'(1)\bar{\gamma}] + \delta(1 - R_i) \frac{\alpha(R_m) + 2\bar{\gamma}}{1 + \delta(R_m - R_i)}\right\} \tag{21}$$

To obtain the integral for the moment about axis 1, substitute (17) and (18) into (15). The resulting expression will again consist of a tilt component and a coning component. The tilt component can be evaluated by moving the θ-independent terms outside of the integral and substituting the appropriate integrals from the appendix, yielding

$$M_{1\epsilon\,\text{tilt}} = -\frac{\pi\gamma R_m^2}{\delta^3 (1 + Z)^2 (1 - R_i)^2} \times [\epsilon(\dot{\psi}_{a\epsilon} - \omega_2) \sin \phi_\epsilon - \dot{\epsilon} \cos \phi_\epsilon] \times \left\{[\alpha(1) - \alpha(R_i) - 3\alpha'(1)\bar{\gamma}] + \delta(1 - R_i) \frac{\alpha(R_m) + 2\bar{\gamma}}{1 + \delta(R_m - R_i)}\right\} \tag{22}$$

The coning term is evaluated as before by substituting for θ_ϵ, applying the angle difference identities, and substituting the applicable integrals from the Appendix. The result is

$$M_{1\epsilon\,\text{coning}} = \frac{2\pi\beta R_m^2}{\delta^3(1+Z)^2(1-R_i)^2} \times \left[\alpha'(1) - \frac{\delta(1-R_i)}{1+\delta(R_m-R_i)}\right] \times [-\epsilon(\dot{\psi}_{a\epsilon}-\omega_2)\sin\phi_\epsilon + \dot{\epsilon}\cos\phi_\epsilon] \quad (23)$$

The integral for the moment about axis 2, $M_{2\epsilon}$, is obtained using (16) and making the same substitutions as for $M_{1\epsilon}$. When the integrals are evaluated, the tilt component of $M_{2\epsilon}$ is zero. The coning term is evaluated as previously, with the result

$$M_{2\epsilon\,\text{coning}} = \frac{2\pi\beta R_m^2}{\delta^3(1+Z)^2(1-R_i)^2} \times \left[\alpha'(1) - \frac{\delta(1-R_i)}{1+\delta(R_m-R_i)}\right] \times [\epsilon(\dot{\psi}_{a\epsilon}-\omega_2)\cos\phi_\epsilon + \dot{\epsilon}\sin\phi_\epsilon] \quad (24)$$

The Rotor Dynamic Coefficients in the Axial and Angular Modes

The general definitions of the stiffness and damping coefficients have the form

$$k_{ij} = -\left.\frac{\partial F_j}{\partial X_i}\right|_{eq}; \quad d_{ij} = -\left.\frac{\partial F_j}{\partial \dot{X}_i}\right|_{eq}$$

respectively, where F_j represents a generalized force acting in degree of freedom j; X_i represents a displacement perturbation occurring in degree of freedom i, and $\dot{X}_i$ represents a velocity perturbation in degree of freedom i. For axial deflections, $i = 3$, and for tilts about axes 1 and 2, i takes the number corresponding to the axis.

The equilibrium configuration is defined to be that in which both seal elements are perpendicular to the seal centerline; in which the centerline clearance, C, takes on its initial value, C_0; in which the magnitude of the eccentricity, ϵ, is zero; and in which the velocities in each degree of freedom are zero. Thus, in the equilibrium configuration all of the position variables (Z, γ_1, γ_2, γ, and ϵ) and their time derivatives are zero, and these zero values are substituted into the evaluations of the partial derivatives to determine the expressions for the stiffness and damping coefficients. The generalized forces can then be obtained from the coefficients using the relation

$$F_j = F_j|_{eq} - \sum_{i=1}^{3} k_{ij}X_i - \sum_{i=1}^{3} d_{ij}\dot{X}_i$$

These generalized forces represent the forces and moments applied to the individual rotors by the fluid film, and each is substituted directly into the equation of motion for the appropriate degree of freedom.

In a concentric analysis, only the axial and tilt degrees of freedom are important. Thus, the rotor dynamic coefficients presented by Wileman and Green (1991) represent only the derivatives of the axial forces and the tilting moments taken with respect to the axial deflection, Z, and the relative tilt between the elements, γ. When these same derivatives are evaluated using only the incremental axial forces and tilting moments which result from eccentricity, the results vanish at equilibrium (Wileman, 1994). Thus, the rotor dynamic coefficients derived by Wileman and Green (1991) are unchanged by the inclusion of eccentricity.

In the eccentric analysis, however, the forces and moments will also vary with the relative eccentricity, ϵ, and additional rotor dynamic coefficients must be obtained to include this dependence in the equations of motion for the axial forces and tilting moments. These coefficients are defined as

$$k_{\epsilon1} = -\left.\frac{\partial M_1}{\partial \epsilon}\right|_{eq}; \quad k_{\epsilon2} = -\left.\frac{\partial M_2}{\partial \epsilon}\right|_{eq}; \quad k_{\epsilon3} = -\left.\frac{\partial F}{\partial \epsilon}\right|_{eq}$$

$$d_{\epsilon1} = -\left.\frac{\partial M_1}{\partial \dot{\epsilon}}\right|_{eq}; \quad d_{\epsilon2} = -\left.\frac{\partial M_2}{\partial \dot{\epsilon}}\right|_{eq}; \quad d_{\epsilon3} = -\left.\frac{\partial F}{\partial \dot{\epsilon}}\right|_{eq} \quad (25)$$

The subscript ϵ indicates that the coefficient represents a change with eccentricity, and the numeral represents the degree of freedom of the generalized force, as before.

Taking the derivatives of the axial force and setting $\gamma = 0$ yields $k_{\epsilon3} = 0$ and $d_{\epsilon3} = 0$. Thus, the axial equations of motion are independent of the radial motion.

When the derivatives of M_1 and M_2 in (25) are evaluated at equilibrium, only the coning terms will remain. Substituting (23) and (24) into the definitions yields

$$k_{\epsilon1} = d_{M\epsilon}(\dot{\psi}_{a\epsilon} - \omega_2)\sin\phi_\epsilon \quad (26a)$$

$$d_{\epsilon1} = -d_{M\epsilon}\cos\phi_\epsilon \quad (26b)$$

$$k_{\epsilon2} = -d_{M\epsilon}(\dot{\psi}_{a\epsilon} - \omega_2)\cos\phi_\epsilon \quad (26c)$$

$$d_{\epsilon2} = -d_{M\epsilon}\sin\phi_\epsilon \quad (26d)$$

where $d_{M\epsilon}$ is a shorthand term defined as

$$d_{M\epsilon} = 2\pi\beta R_m^2 G_0 \quad (27)$$

G_0 is defined as

$$G_0 = \frac{\ln[1+\beta(1-R_i)] - \dfrac{2\beta(1-R_i)}{2+\beta(1-R_i)}}{\beta^3(1-R_i)^2}$$

The linearized moments acting upon rotor 2 are then expressed in the fluid film system as

$$M_1 = -d_{M\epsilon}\epsilon(\dot{\psi}_{a\epsilon} - \omega_2)\sin\phi_\epsilon + d_{M\epsilon}\dot{\epsilon}\cos\phi_\epsilon \quad (28a)$$

$$M_2 = d_{M\epsilon}\epsilon(\dot{\psi}_{a\epsilon} - \omega_2)\cos\phi_\epsilon + d_{M\epsilon}\dot{\epsilon}\sin\phi_\epsilon \quad (28b)$$

The moments acting upon rotor 1 will be the negatives of these.

The Radial Forces

Etsion and Sharoni (1980) showed that radial forces will result from misalignment and coning in concentric seals. In a concentric system these forces are usually of interest only because of the loads they apply to the bearings in the system. If eccentric motion is possible, however, each rotor will have two additional degrees of freedom, each requiring its own equation of motion. In order to incorporate the radial forces applied by the fluid film into these equations, it will be necessary to obtain additional rotor dynamic coefficients.

In contrast to the axial force and tilting moments, the radial forces in a seal depend upon shear stresses rather than normal stresses; it is not necessary to solve the Reynolds equation to obtain these forces. Rather, we simply use the velocities of the seal surfaces as boundary conditions and assume Couette flow within the fluid film to obtain the shear stresses. There will actually be a small Poiseuille component contributing to the shear stresses, but Etsion and Sharoni (1980) have shown that this effect is negligible in narrow seals.

The radial forces in an eccentric FMRR seal are more complex than those obtained by Etsion and Sharoni for the concentric FMS configuration. They consist of two separate effects. The first component results from the interaction between the rotation of the seal seat and the radial distance between the centers of the ring and the seat. This component of the total radial force exists even when the eccentricity of the system is constant; that is, if the element centers are fixed in space. It depends only upon the velocity components contained in (5) and (6). We shall refer to this component as the displacement force.

The second component results from the relative motion of the centers of the two elements. This component of the radial force exists only if the shafts are flexible and is not directly affected by the shaft rotation. It depends upon the relative velocity between the centers of rotation defined by (3). This component will be referred to as the velocity force.

To simplify the derivation, we shall examine the displacement and velocity components separately and add the final results to obtain the total radial force.

The Displacement Force

Because the fluid film is very thin, we make the usual assumption that the velocities in the axial direction are negligible. Using this assumption, and assuming Couette flow, the shear stresses in the radial and circumferential directions, respectively, are defined by

$$\tau_{\epsilon r} = \mu \frac{\partial v_{\epsilon r}}{\partial z}; \quad \tau_{\epsilon \theta} = \mu \frac{\partial v_{\epsilon \theta}}{\partial z} \tag{29}$$

The velocity derivatives in (29) are obtained by computing the difference between the two velocities of (5) and (6) and dividing by the clearance, h. Thus,

$$\tau_{\epsilon r} = -\frac{\mu \epsilon \omega_2 \sin \theta_\epsilon}{h} \tag{30a}$$

$$\tau_{\epsilon \theta} = \frac{\mu r(\omega_2 - \omega_1)}{h} - \frac{\mu \epsilon \omega_2 \cos \theta_\epsilon}{h} \tag{30b}$$

To obtain the total displacement force, we shall divide the shear stresses in (30) into concentric and eccentric terms. The first term of (30b) represents stresses which occur even when the seal is concentric. This concentric term will produce a radial force only in the presence of relative misalignment, as it integrates to zero over the seal ring surface if the faces are parallel.

For faces which are both coned and misaligned, the film thickness in the seal is described by (10). The concentric term of (30b) is symmetric about the $\hat{2}$ axis, and will produce a radial force only in the direction parallel to the $\hat{1}$ axis. This force, denoted F, can be obtained by integration directly in the fluid film (123) system, then resolved into components, denoted F_1 and F_2, in the eccentric $(123)_\epsilon$ system. Thus,

$$F = -2\mu r_m^2(\omega_2 - \omega_1) \int_0^\pi \int_{r_i}^{r_o} \frac{\cos \theta}{h} dr d\theta \tag{31}$$

The integral is evaluated by Etsion and Sharoni (1980). When the result is resolved into components in the $(123)_\epsilon$ system, the radial forces become

$$F_2 = -\mu r_m^2(\omega_2 - \omega_1)\pi \left(\frac{r_o}{C_0}\right)^2 (1 - R_i^2) W_0 \gamma \sin \phi_\epsilon \tag{32a}$$

$$F_1 = \mu r_m^2(\omega_2 - \omega_1)\pi \left(\frac{r_o}{C_0}\right)^2 (1 - R_i^2) W_0 \gamma \cos \phi_\epsilon \tag{32b}$$

where W_0 is defined using normalized variables as

$$W_0 = \frac{\ln [1 + \beta(1 - R_i)] - \dfrac{\beta(1 - R_i)(1 - \beta R_i)}{1 + \beta(1 - R_i)}}{\beta^2(1 - R_i^2)}$$

These forces result only from relative tilt and are independent of the eccentricity. This component of the displacement force will exist even in a seal with concentric shafts if the two elements are misaligned.

The remaining terms of the shear stresses in (30) result from the eccentric displacement. The resulting forces are obtained by resolving the shear stress components in the cylindrical system into components parallel to the $\hat{1}_\epsilon$ and $\hat{2}_\epsilon$ axes and integrating these over the area of the sealing faces. The forces are defined as

$$F_2 = \int_0^{2\pi} \int_{r_i}^{r_o} (\tau_{\epsilon r} \cos \theta_\epsilon - \tau_{\epsilon \theta} \sin \theta_\epsilon) r dr d\theta \tag{33a}$$

$$F_1 = \int_0^{2\pi} \int_{r_i}^{r_o} (-\tau_{\epsilon r} \sin \theta_\epsilon - \tau_{\epsilon \theta} \cos \theta_\epsilon) r dr d\theta \tag{33b}$$

When the eccentric terms of the shear stresses in (30) are substituted into (33), the two terms in the integrand of (33a) add out. Thus, F_2 will be zero.

When integrating to obtain F_1, the relative tilt between the two elements leads only to second order effects in the resulting force. This can be shown by comparing the force which results from parallel faces to that which results when a relative misalignment exists. For parallel faces the film thickness, $h = C$, is constant, and the radial force which results from (33b) is

$$F_1 = \frac{2\pi \mu \epsilon \omega_2 r_m (r_o - r_i)}{C} \tag{34}$$

When the faces are tilted (but the coning is not included), the film thickness is defined by

$$h = C + \gamma r \cos \theta \tag{35}$$

Substituting the eccentric shear stresses into (33b), the expression for F_1 becomes

$$F_1 = \mu \epsilon \omega_2 r_m \int_0^{2\pi} \int_{r_i}^{r_o} \frac{dr d\theta}{h} \tag{36}$$

The integration over r is performed by substituting $dh = \gamma \cos \theta \, dr$. Thus,

$$\int_{r_i}^{r_o} \frac{dr}{h} = \frac{1}{\gamma \cos \theta} \int_{h_i}^{h_o} \frac{dh}{h} = \frac{\log \left(\dfrac{h_o}{h_i}\right)}{\gamma \cos \theta} \tag{37}$$

In order to evaluate (36), the logarithms in (37) must be expanded as a series. Separate the logarithm of the quotient into two terms by dividing both the numerator and the denominator of the argument by C_0 to obtain $\log (h_o/C_0) - \log (h_i/C_0)$. Then, when γ is small, we can use the approximation

$$\log \left(\frac{h_o}{C_0}\right) = \log \left[1 + \frac{\gamma r_o \cos \theta}{C_0}\right] \approx \frac{\gamma r_o \cos \theta}{C_0} - \frac{\gamma^2 r_o^2 \cos^2 \theta}{2C_0^2} + \frac{\gamma^3 r_o^3 \cos^3 \theta}{3C_0^3} \tag{38}$$

Substituting this into the first term of (36) yields

$$\int_0^{2\pi} \frac{\log\left(\frac{h_o}{C}\right)}{\gamma \cos\theta} d\theta = \left[2 + \frac{1}{3}\left(\frac{\gamma r_o}{C_0}\right)^2\right] \frac{\pi r_o}{C_0}$$

Performing a similar substitution for the second term yields, after some simplification,

$$F_1 = \frac{\pi\mu\epsilon\omega_2 r_m(r_o - r_i)}{C_0}\left[2 + \gamma^2\left(\frac{r_o^2 + r_o r_i + r_i^2}{3C_0^2}\right)\right] \quad (39)$$

Comparing (39) with (34), it is clear that the tilt contributes only a second order term to the radial force, and this term will vanish during the linearization process by which rotor dynamic coefficients are obtained. Thus, for the remainder of the radial force analysis we will neglect the tilt. The effect of this approximation upon the accuracy of the final result will depend upon the actual magnitude of the tilt in the system. For simplicity, the preceding analysis of the effects of tilt has not included the effects of coning, but coning will generally further reduce the effects of tilt.

Neglecting the tilt (but not the coning) greatly simplifies the derivation of the radial force. The expression for the film thickness becomes

$$h = C + \beta(r - r_i) \quad (40)$$

The integration over r in (33b) takes the form

$$\int_{r_i}^{r_o} \frac{dr}{h} = \frac{1}{\beta}\int_{h_i}^{h_o} \frac{dh}{h} = \frac{\log\left(\frac{h_o}{h_i}\right)}{\beta}$$

$$= \frac{1}{\beta}\log\left[1 + \frac{\beta r_o(1 - R_i)}{C}\right]$$

Substituting this into (33b) and evaluating the θ integration yields the radial force which results from a fixed eccentricity between the two elements:

$$F_1 = \frac{2\pi\mu\epsilon\omega_2 r_m \log\left[1 + \frac{\beta r_o(1 - R_i)}{C}\right]}{\beta} \quad (41)$$

In the limit as β goes to zero (i.e., flat faces), (41) approaches (34).

It is interesting to note that the rotation of the seal ring does not contribute to the radial force. Only the angular velocity of the seat affects the force. The sealing dam is bounded by the seal ring, and the center of the ring is the origin of the system used to integrate the shear stresses to obtain the radial forces. Since the shear stresses which result from the motion of the ring are axisymmetric with respect to the eccentric system, they will integrate to zero over the complete circumference. The sealing dam is not axisymmetric with respect to the center of the seal seat, however. Thus, the shear stresses resulting from the seat motion integrate to produce net radial forces.

The Velocity Force

If the shafts can deflect radially from their equilibrium positions, then the radial motion and the orbit of the center of rotation will cause a uniform velocity over the entire sealing face. The velocity of the center of element 2 with respect to that of element 1 (i.e., the relative velocity) is expressed in Eq. (3), and this relative velocity will produce a linear velocity profile between the two elements.

The derivation of the force which results from this velocity resembles the analysis previously performed for the eccentric components of the displacement force. Substituting the velocity components in (3) into the shear stress definitions in (29), these new shear stresses are

$$\tau_{\epsilon r} = \mu\frac{\partial v_{\epsilon r}}{\partial z} = \frac{\mu}{h}(\epsilon\dot{\psi}_{a\epsilon}\sin\theta_\epsilon + \dot{\epsilon}\cos\theta_\epsilon)$$

$$\tau_{\epsilon\theta} = \mu\frac{\partial v_{\epsilon\theta}}{\partial z} = \frac{\mu}{h}(\epsilon\dot{\psi}_{a\epsilon}\cos\theta_\epsilon - \dot{\epsilon}\sin\theta_\epsilon) \quad (42)$$

Analysis of the effects of tilt for the velocity force resembles that already performed for the displacement force. Thus, as before, the effects of tilt are second order and can be neglected so that h is as defined in (40). The shear stresses are integrated using (33), as before, to determine their contributions to the radial forces. For F_2 the first term in each of the shear stresses adds out of the integrand, leaving only the second term. Thus,

$$F_2 = \mu\dot{\epsilon}r_m\int_0^{2\pi}\int_{r_i}^{r_o}\frac{drd\theta}{h}$$

$$= \frac{2\pi\mu\dot{\epsilon}r_m\log\left[1 + \frac{\beta r_o(1 - R_i)}{C}\right]}{\beta} \quad (43)$$

For F_1 the second term in each of the shear stresses adds out, and following a similar development yields

$$F_1 = -\frac{2\pi\mu\epsilon\dot{\psi}_{a\epsilon}r_m\log\left[1 + \frac{\beta r_o(1 - R_i)}{C}\right]}{\beta} \quad (44)$$

The Rotor Dynamic Coefficients in the Radial Modes

The total radial force components are obtained by summing the velocity forces, (43) and (44), and the concentric (32) and eccentric (41) components of the displacement force. The resulting forces are then normalized using the definitions (12) with the result

$$F_2 = \frac{\pi\dot{\epsilon}R_m\log[1 + \beta(1 - R_i)]}{3\beta(1 - R_i)^2} - \frac{\pi\gamma\left(\frac{C_0}{r_o}\right)(\omega_2 - \omega_1)}{6} W_0\sin\phi_\epsilon \quad (45a)$$

$$F_1 = \frac{\pi\epsilon(\omega_2 - \dot{\psi}_{a\epsilon})R_m\log[1 + \beta(1 - R_i)]}{3\beta(1 - R_i)^2} + \frac{\pi\gamma\left(\frac{C_0}{r_o}\right)(\omega_2 - \omega_1)}{6} W_0\cos\phi_\epsilon \quad (45b)$$

Because the eccentricity in the relative system is defined by the single variable, ϵ, there will be only two stiffness and two damping coefficients for the radial forces relating the radial forces to the radial deflections and velocities. These are defined as

$$k_{\epsilon 2r} = -\frac{\partial F_2}{\partial\epsilon}\bigg|_{eq}; \quad k_{\epsilon 1r} = -\frac{\partial F_1}{\partial\epsilon}\bigg|_{eq}$$

$$d_{\epsilon 2r} = -\frac{\partial F_2}{\partial\dot{\epsilon}}\bigg|_{eq}; \quad d_{\epsilon 1r} = -\frac{\partial F_1}{\partial\dot{\epsilon}}\bigg|_{eq}$$

where the r in the subscript distinguishes the coefficients in the radial modes from those derived previously for the angular modes. Because F_1 and F_2 also depend upon the relative tilt, γ, there will be two additional coefficients defined as

$$k_{\gamma 1r} = -\left.\frac{\partial F_1}{\partial \gamma}\right|_{eq}; \quad k_{\gamma 2r} = -\left.\frac{\partial F_2}{\partial \gamma}\right|_{eq}$$

Since the derivatives of the radial forces with respect to the axial deflection will vanish at equilibrium, the axial modes remain completely decoupled from both the angular and radial modes.

When (45) is substituted into these definitions, $k_{\epsilon 2r}$ and $d_{\epsilon 1r}$ are zero. The remaining rotor dynamic coefficients in the eccentric system are

$$k_{\epsilon 1r} = -\frac{\pi(\omega_2 - \dot{\psi}_{a\epsilon})R_m \log[1 + \beta(1 - R_i)]}{3\beta(1 - R_i)^2} \quad (46a)$$

$$d_{\epsilon 2r} = -\frac{\pi R_m \log[1 + \beta(1 - R_i)]}{3\beta(1 - R_i)^2} \quad (46b)$$

$$k_{\gamma 1r} = -\frac{\pi\left(\frac{C_0}{r_o}\right)(\omega_2 - \omega_1)}{6} W_0 \cos \phi_\epsilon \quad (46c)$$

$$k_{\gamma 2r} = \frac{\pi\left(\frac{C_0}{r_o}\right)(\omega_2 - \omega_1)}{6} W_0 \sin \phi_\epsilon \quad (46d)$$

Note that the stiffness term $k_{\epsilon 1r}$ is related to the damping term $d_{\epsilon 2r}$ by

$$k_{\epsilon 1r} = d_{\epsilon 2r}(\omega_2 - \dot{\psi}_{a\epsilon})$$

Note also that the two tilt coefficients share a common factor which it will be convenient to divide out. Define

$$k_\gamma = \frac{\pi\left(\frac{C_0}{r_o}\right)(\omega_2 - \omega_1)W_0}{6}$$

Substituting into (46) yields

$$k_{\gamma 1r} = -k_\gamma \cos \phi_\epsilon; \quad k_{\gamma 2r} = k_\gamma \sin \phi_\epsilon$$

Since only one of the damping coefficients is nonzero, we shall refer to $d_{\epsilon 2}$ as d_ϵ. Using all of the simplifications, the radial forces acting upon element 2 in the eccentric system become

$$F_{1\epsilon} = -d_\epsilon(\omega_2 - \dot{\psi}_{a\epsilon})\epsilon - k_\gamma \cos \phi_\epsilon \gamma \quad (47a)$$

$$F_{2\epsilon} = -d_\epsilon \dot{\epsilon} + k_\gamma \sin \phi_\epsilon \gamma \quad (47b)$$

The forces acting upon element 1 will be the negatives of these.

Conclusions

This work provides the tools necessary to incorporate the effects of eccentricity into the dynamic analysis of a mechanical seal. The kinematic model used to obtain the rotor dynamic coefficients can also be used to describe the radial motion of the rotors in the equations of motion.

Including eccentricity has no effect upon the change in the axial force and tilting moments with respect to either tilt or axial deflection. Thus, the rotor dynamic coefficients derived by Wileman and Green (1991) for the concentric case remain unchanged in the eccentric analysis. Further, the effect of radial deflections upon the axial force is negligible, as is the effect of the axial deflection upon the tilting moments and radial forces. Thus, the axial mode remains completely decoupled from all of the others.

The tilting moments are shown to be dependent upon the radial deflections and velocities, leading to stiffness and damping coefficients which will couple the tilting modes with the radial modes. These rotor dynamic coefficients must be incorporated into the equations of motion for the angular modes.

The forces in the radial direction are obtained and are shown to be dependent upon both the radial and the angular deflections of the rotors. The rotor dynamic coefficients obtained from these forces can be used to obtain two additional equations of motion for each element. Each of these equations will represent a force balance in one of the radial directions and will contain coupling between the radial and tilting modes. The equations of motion can ultimately be solved for the dynamic response. Only a complete solution will reveal the relative significance of the responses to the axial, tilt, and eccentricity effects. The importance of eccentricity in the analysis will depend particularly upon the shaft inertia and flexibility and the magnitudes of the eccentric displacements and velocities. In many cases it may be possible to neglect eccentricity entirely, but such a judgement can only be made by comparing the forces and moments which occur in a particular seal design.

The advantage of using linearized rotor dynamic coefficients is that it allows direct solution for the dynamic response, rather than the iterative approach necessary when the full nonlinear equations are used. Furthermore, the assumptions inherent in the linearized solution are very representative of the conditions which exist in real seal applications.

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APPENDIX

Integrals Involving $T(\theta)$

The following integrals evaluated by Wileman (1994) are used to obtain the axial force and the tilting moments resulting from the hydrodynamic pressure field:

$$\int_0^{2\pi} T(\theta)d\theta = 2\pi\left[\frac{2\alpha'(1)}{\delta^3} - \frac{2(1-R_i)}{\delta^2[1+\delta(R_m-R_i)]}\right] \quad (48a)$$

$$\int_0^{2\pi} T(\theta)\cos\theta d\theta = \frac{2\pi}{\delta^3}[\alpha(1) - \alpha(R_i) - 3\alpha'(1)\bar{\gamma}] + 2\pi(1-R_i)\frac{\alpha(R_m)+2\bar{\gamma}}{\delta^2[1+\delta(R_m-R_i)]} \quad (48b)$$

$$\int_0^{2\pi} T(\theta)\sin\theta d\theta = 0 \quad (48c)$$

$$\int_0^{2\pi} T(\theta)\cos\theta\sin\theta d\theta = 0 \quad (48d)$$

$$\int_0^{2\pi} T(\theta)\cos^2\theta d\theta = \pi\left[\frac{2\alpha'(1)}{\delta^3} - \frac{2(1-R_i)}{\delta^2[1+\delta(R_m-R_i)]}\right] \quad (48e)$$

$$\int_0^{2\pi} T(\theta)\sin^2\theta d\theta = \pi\left[\frac{2\alpha'(1)}{\delta^3} - \frac{2(1-R_i)}{\delta^2[1+\delta(R_m-R_i)]}\right] \quad (48f)$$