On the Stability of Gas Lubricated Triboelements Using the Step Jump Method

B. Miller

I. Green

George W. Woodruff School
of Mechanical Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332-0405

The step jump method was developed approximately three decades ago to help determine the stability of gas lubricated triboelements. In the approach, the force contribution from the gas layer is characterized by its step response, which is the transient force response resulting from pressure diffusion in the gas film after a step increase in film thickness. The procedure is broadened by implementing Duhamel's theorem to yield the system characteristic equation. Since its inception in the literature, the step response has been approximated in the equations of motion using a series of Laguerre polynomials, which allows for a closed form analysis. This paper will prove that using Laguerre polynomials can violate the second law of thermodynamics, and a test case will show that stability results predicted by this approach can be inaccurate. It will be proven that a mathematical correlation exists between the dynamic behavior of the gas film and the dynamic behavior of a linear viscoelastic medium. This correlation is advantageous since much of the viscoelastic theory can be applied to the dynamic analysis of gas lubricated triboelements.

Introduction

Analyzing the dynamic behavior of typical gas lubricated triboelements is difficult and is usually a computationally intensive endeavor. The force from the gas film is included in the equations of motion as an external force and is formulated by integrating the pressure over the subjected surface area. The pressure distribution is obtained from the unsteady nonlinear Reynolds equation.

Two techniques are commonly used to analyze the system dynamics. The first method, called the time-transient method, directly solves the equations of motion and the Reynolds equation using a numerical scheme (Castelli and McCabe, 1967). This technique requires a solution of the Reynolds equation at each time interval. The resulting force is then calculated and entered directly into the equations of motion. Finally, these equations are integrated numerically yielding full time histories for each degree of freedom. The second method, called the step jump approach (Elrod et al., 1967 and Shapiro and Colsher, 1970), makes the assumption that the gas film is linear in response to successive, small step increases in the various degrees of freedom, presuming that the motion is about an equilibrium state. In the approach as it appears in literature, the step response is approximated by a series of Laguerre polynomials, thus providing an analytic expression for the characteristic equation. Pertinent dynamic parameters are then found by solving for the roots of the characteristic equation. This method has been largely adopted by many researchers (e.g., Kazimierski and Jarzecki, 1979; Etsion and Green, 1981; Sela and Blech, 1991), and the method detailed by Elrod et al. (1967) has been reproduced in the textbook by Gross (1980).

The techniques mentioned above are two distinctly different methods, and each has its merits and faults. One advantage of the time-transient method is that it gives a large amount of detailed information about each degree of freedom. However, it also requires a large number of computer calculations since the Reynolds equation must be solved at each time step. This can be time consuming if the model has many degrees of freedom. In contrast, a significant merit of the step jump approach is that it offers a substantial savings in computing time. In this method, the Reynolds equation is only solved during the process of generating the step response for each system degree of freedom, and then the analysis continues using analytic functions and yielding closed-form solutions.

Overall, the step jump approach shows promise for providing valuable information concerning the dynamics of gas lubricated triboelements. However, this paper will prove that approximating the step response of the fluid film by a series of Laguerre polynomials, though mathematically elegant, is thermodynamically inappropriate in some cases and can render an invalid film representation. Consequently, results presented in the literature, which relied on the Laguerre polynomial formulation, may not be credible.

The purpose of this work is to determine whether using Laguerre polynomials in the step jump method is thermodynamically admissible and to establish a relationship between the dynamic properties of linear viscoelastic materials and gas films in triboelements. Furthermore, it will be shown that the step response of the gas film strongly resembles the relaxation characteristics of common linear viscoelastic materials and the behavior of the fluid region can be modeled like a linear viscoelastic medium. For simplicity, all of the theory and examples presented here will relate to a gas lubricated, thrust slider bearing.

Thrust Slider Bearing

A gas lubricated, thrust slider bearing of infinite width (into the page) is shown in Fig. 1. The bearing has a very simple geometry and is assumed to have just one degree of freedom, which is motion in the y-direction. The behavior of the gas region is analyzed using the isothermal form of the unsteady Reynolds equation:

$$\frac{\partial}{\partial x}\left(ph^3\frac{\partial p}{\partial x}\right) = \Lambda\left[\frac{\partial(ph)}{\partial x} + \frac{\partial(ph)}{\partial t}\right] \quad (1)$$

where

$$p = \frac{p^*}{P_a} \quad h = \frac{h^*}{h^*_{o,\mathrm{eq}}} \quad x = \frac{x^*}{B} \quad t = \frac{V}{2B}t^* \quad \Lambda = \frac{6\mu VB}{P_a h^{*2}_{o,\mathrm{eq}}} \quad (2)$$

Contributed by the Tribology Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS and presented at the ASME/STLE Joint Tribology Conference, San Francisco, Calif., October 13–17, 1996. Manuscript received by the Tribology Division December 18, 1995: revised manuscript received June 6, 1996. Paper No. 96-Trib-12. Associate Technical Editor: J. Frene.

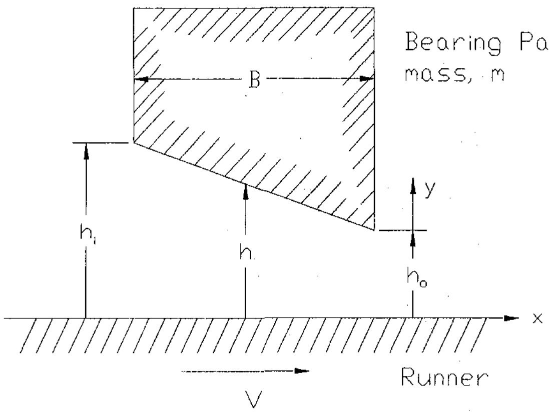

Fig. 1 Schematic of thrust slider bearing

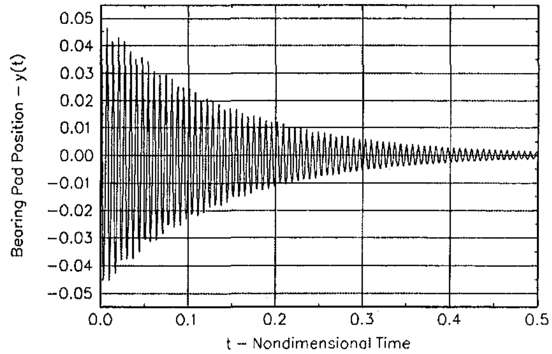

Fig. 2 Transient response of bearing pad

Note that the gas film thickness is nondimensionalized by the minimum film thickness at the equilibrium state, which occurs at the outlet region. This is also the value used to formulate the nondimensional compressibility number.

The bearing discussed in this paper has the following dimensions and operating conditions:

Minimum film thickness,	$h^*_{o,\mathrm{eq}} = 0.5\ \mu\mathrm{m}$
Inlet/outlet region ratio,	$h^*_{i,\mathrm{eq}}/h^*_{o,\mathrm{eq}} = 1.8$
Velocity,	$V = 0.56$ m/s
Bearing length,	$B = 4\ (10)^{-3}$ m
Viscosity of air,	$\mu = 1.86\ (10)^{-5}\ \mathrm{Ns/m^2}$
Ambient pressure,	$P_a = (10)^5\ \mathrm{N/m^2}$
Bearing pad mass per unit width,	$m^* = 0.16$ kg/m
Nondimensional bearing pad mass,	$m = 10^{-6}$
Compressibility number,	$\Lambda = 10.0$

To verify that the system is stable, a time-transient dynamic analysis was performed to find the bearing response to an initial small disturbance, $\Delta y^* = 2.5\ (10)^{-8}$ m, in the film thickness. The coupled unsteady Reynolds equation and the equation of motion were simultaneously integrated in time, and a time history of the bearing displacement is shown in Fig. 2. Clearly, the amplitude envelope is continuously decaying, which indicates that the bearing system is stable, and analyzing this bearing by the step jump method ought to produce the same result. The outline of the step jump method that follows is a modification of the method by Elrod et al. (1967).

Step Jump Approach

In the step jump approach, the dynamic behavior of the gas film is characterized by its response to perturbations in each degree of freedom. The primary assumption (Elrod et al., 1967) is that the gas film response is linear, which suggests that superposition of inputs leads to superposition of outputs. This assumption imposes stringent restrictions on applications of this method, and careful attention must be paid to ensure that this assumption is not violated.

In linear systems, if a general disturbance is considered a superposition of small step disturbances, then the total system response is a superposition of responses to small step disturbances. As a result, applying Duhamel's theorem directly results in a solution for the total force response of the fluid film. This theorem states that the film load carrying capacity, $f(t)$, can be found by summing the individual step responses, $k(t)$, to each step input, Δy, over the full duration of time. In equation form, this yields:

Nomenclature

A_n = nth Laguerre coefficient
B = Slider bearing length in x-direction
$E(t)$ = viscoelastic relaxation modulus
$f^*(t)$ = gas film force response
$f(t)$ = nondimensional gas film force, $f^*(t)/P_a(B\cdot 1)$
$F(s)$ = Laplace transform of $f(t)$
h^* = film thickness
$h^*_{o,\mathrm{eq}}$ = minimum film thickness at equilibrium
h = nondimensional film thickness, $h^*/h^*_{o,\mathrm{eq}}$
$k^*(t)$ = gas film step response, $-\Delta f^*(t)/\Delta y^*$
$k(t)$ = nondimensional gas film step response (relaxation function), $k^*(t)h^*_{o,\mathrm{eq}}/P_a(B\cdot 1)$
k_g = linear spring modulus
$K(s)$ = Laplace transform of $k(t)$
$K'(\omega)$ = loss factor
L_n = Laguerre polynomial of the nth order
L^1_n = first generalized Laguerre polynomial of the nth order
m^* = mass of the bearing pad per unit width
m = nondimensional mass, $m^*V^2h^*_{o,\mathrm{eq}}/4B^3P_a$
N = highest order of Laguerre polynomial approximation
P = number of degrees of freedom
P_a = ambient pressure
p^* = pressure
p = nondimensional pressure, p^*/P_a
s = Laplace variable
t^* = time
t = nondimensional time, $Vt^*/2B$
x^* = general coordinate
x = nondimensional variable, x^*/B
y^* = general coordinate
y = nondimensional displacement variable, $y^*/h^*_{o,\mathrm{eq}}$
Δy^* = step jump in y-direction
Δy = nondimensional step-jump, $\Delta y^*/h^*_{o,\mathrm{eq}}$
$Y(s)$ = Laplace transform of $y(t)$
V = velocity of runner in x-direction
α = attenuation factor
$\epsilon(t)$ = strain
Λ = nondimensional compressibility number, $6\mu VB/P_a h^{*2}_{o,\mathrm{eq}}$
μ = gas absolute viscosity
ν = complex exponential
$\sigma(t)$ = stress
ω^* = frequency variable
ω = frequency variable, $2B\omega^*/V$

Subscripts and Superscripts

crit = critical value at stability threshold
eq = equilibrium state
i = inlet region
o = outlet region
$*$ = dimensional variable
∞ = steady state; long time

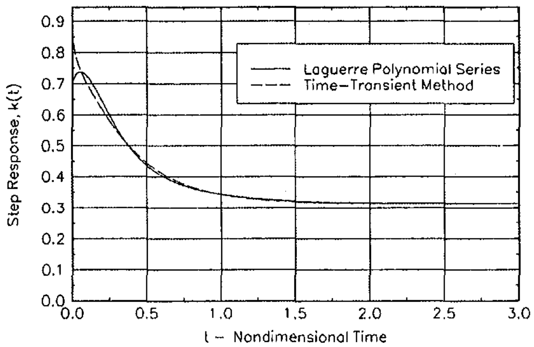

Fig. 3 Approximate (Laguerre polynomials) and actual step response curves, $k(t)$

$$f(t) = y(0)k(t) + \sum_n (\Delta y)_n k(t - n\Delta\tau) \quad (3)$$

where

$f(t)$ = force response of the fluid film at time t
$y(t)$ = displacement of the bearing pad at time t
$k(t - n\Delta\tau)$ = step response of the fluid film at time t produced by a unit step increase in the film thickness at time $n\Delta\tau$

Note that $f(t)$ is considered positive when the net force produced by the gas film is pointed away from the bearing pad and into the gas film.

As the time increments ($\Delta\tau$) between step inputs become smaller and approach zero, Eq. (3), in the limit, becomes Duhamel's integral,

$$f(t) = y(0)k(t) + \int_0^t \dot{y}(\tau)k(t - \tau)d\tau \quad (4)$$

where $\dot{y} = d[y(\tau)]/d\tau$. In the literature (e.g., Elrod et al., 1967), the step response produced by a unit step increase in a degree of freedom is usually denoted by $H(t)$. However, in this work it will be replaced by $k(t)$, which will sometimes be referred to as the stiffness of the fluid film since it represents the transient force of the fluid film per unit displacement. The relationship between $k(t)$ and $H(t)$ is simply $k(t) = -H(t)$. A positive $k(t)$ will produce a restoring force, $f(t)$, on the bearing.

Response of the Gas to a Step Jump. In this work, the step response was numerically generated by a method similar to the time-transient dynamic analysis technique mentioned earlier. The bearing pad is initially assumed to be at an equilibrium height corresponding to the steady-state value where the force of the fluid film, f_{eq}, is in balance with the other applied forces, e.g., the weight of the bearing pad. After the initial steady-state pressure profile is calculated, the bearing is given a jump in height, Δy, and held there. Since isothermal conditions are assumed in the Reynolds equation, the jump is also assumed isothermal. Therefore, the product of pressure and height, ph, remains constant allowing the new pressure profile to be calculated at the instant of the jump, $t = 0$. Then, for $t > 0$, the pressure distribution is computed from the unsteady Reynolds equation, allowing it to diffuse to a new steady state condition. During the diffusion process a force, $f(t)$, is calculated at each time step, along with the time-dependent step response, $k(t)$, as defined by

$$k(t) = -\frac{f(t) - f_{eq}}{\Delta y} \quad (5)$$

A typical response curve for a step jump in the bearing film thickness is given in Fig. 3 (see the dashed line).

Analytic Representation Using Laguerre Polynomials. The next stage in the step jump method as outlined by Elrod et al. (1967) is to curve fit the step response with a series of Laguerre polynomials,

$$k(t) = k(\infty) + \sum_{n=0}^{N} A_n L_n(\alpha t)e^{-\alpha t} \quad (6)$$

Letting N approach infinity would render an exact formulation for $k(t)$. However, for numerical considerations N must be finite, and Eq. (6) is, therefore, only an approximation of the step response.

Using a series of Laguerre polynomials is an elegant technique because the polynomials are orthogonal in the interval from zero to infinity with an exponential weighting function. As a result, the Laguerre coefficients are simply found by the relation

$$A_n = \alpha \int_0^\infty L_n(\alpha t)[k(t) - k(\infty)]dt \quad (7)$$

A consequence of the orthogonality relation is that the Laguerre coefficients are independent of the number of polynomials employed in the approximation. Since $k(t)$ is normally available in a tabular form, the Laguerre coefficients are typically calculated by Eq. (7) using numerical integration. It should be noted that the decay factor, α, is a positive parameter that needs to be selected. Some values of α will cause the coefficients of the Laguerre polynomial series to approach zero more rapidly. However, this has to be determined numerically by an arduous trial and error method.

By replacing $k(t)$ in Eq. (4) with its approximation, Eq. (6), a strictly analytic equation is formulated,

$$f(t) = y(0)[k(t) - k(\infty)] + k(\infty)y(t) + \int_0^t \dot{y}(t - \tau) \sum_{n=0}^{N} A_n L_n(\alpha\tau)e^{-\alpha\tau}d\tau \quad (8)$$

Next, the system is assumed to be responding to a small harmonic perturbation about an equilibrium position in the form of

$$y(t) = \bar{y}e^{\nu t} \quad (9)$$

where ν is a complex variable. Substituting Eq. (9) into Eq. (8) yields

$$f(t) = y(0)[k(t) - k(\infty)] + \left[k(\infty) + \frac{\nu}{\alpha}\sum_{n=0}^{N} A_n \int_0^t L_n(\alpha\tau)e^{-(\nu/\alpha+1)\alpha\tau}d(\alpha\tau)\right]\bar{y}e^{\nu t} \quad (10)$$

Now, letting the upper bound in the integral of Eq. (10) approach infinity to account for the step jump response at its completion time, the integral becomes a Laplace transform, giving

$$\int_0^\infty L_n(\psi)e^{-(\nu/\alpha+1)\psi}d\psi = \frac{\alpha}{\nu}\zeta^{n+1} \quad (11)$$

with the following parameters defined:

$$\psi = \alpha\tau \quad \text{and} \quad \zeta = \frac{\nu}{\nu + \alpha} \quad (12)$$

Substituting the result of Eq. (11) into Eq. (10) gives, finally,

$$f(t) = [k(\infty) + \sum_{n=1}^{N+1} A_{(n-1)}\zeta^n]\bar{y}e^{\nu t} \quad (13)$$

Stability Analysis Using Laguerre Polynomials. The equation of motion for the slider bearing in Fig. 1 is

$$m\ddot{y}(t) = -f(t) \quad (14)$$

where m is the nondimensional mass of the bearing, y is the nondimensional displacement variable, and negative f is the net resultant nondimensional force acting on the bearing pad. To determine the stability of the system, a small harmonic perturbation about an equilibrium position is applied. After substituting Eq. (13) into Eq. (14) and using the harmonic perturbation form of Eq. (9), the resulting equation is

$$[m\nu^2 + k(\infty) + \sum_{n=1}^{N+1} A_{(n-1)}\zeta^n]\bar{y}e^{\nu t} = 0 \quad (15)$$

This equation should hold for any nontrivial solution at any time, t. Thus, the bracketed expression, when set equal to zero, is recognized as the characteristic equation. Recalling the definitions in Eq. (12) and arbitrarily choosing $N = 3$, the approximate characteristic equation is written as

$$m\nu^2 + k(\infty) + A_0\frac{\nu}{\nu+\alpha} + A_1\left(\frac{\nu}{\nu+\alpha}\right)^2 + A_2\left(\frac{\nu}{\nu+\alpha}\right)^3 + A_3\left(\frac{\nu}{\nu+\alpha}\right)^4 = 0 \quad (16)$$

Once the characteristic equation is found, determining information about the stability of the system is, in theory, straightforward.

Since the bearing system in question is assumed to be one degree of freedom, the characteristic equation is naturally expected to be second order and have two roots. However, Eq. (16) will have more than two roots. From a physical standpoint, the extra roots are a result of the way the gas film is modeled. The Laguerre polynomial expansion in Eq. (6) represents a dynamic model in which the gas film is replaced by a complex set of springs and dampers. These spring and damper elements add extraneous degrees of freedom to the system (Szumski, 1993). As a result, the approximate characteristic equation will have $P(N + 3)$ roots, where P is the number of degrees of freedom assumed for the bearing, and N is the highest order of Laguerre polynomial used in the approximate expansion.

The roots of the characteristic equation give information concerning the response envelope and the frequency of oscillation for each mode of vibration. If any roots of the approximate characteristic equation have positive real parts then the system is unstable. Likewise, if all the roots have negative real parts, then the system is stable. It must be stated that obtaining the roots of a characteristic equation of order $P(N + 3)$ becomes more difficult with the increase of P or N. Since a relatively large N is typically necessary to fit $k(t)$ adequately and very few systems can realistically be modeled by a low number of degrees of freedom, P, then a solution may not always be feasible.

For this example, let $P = 1$, $N = 3$, and $\alpha = 5.0$. The following values are then calculated using Eq. (7): $A_0 = 0.9157$, $A_1 = -0.6677$, $A_2 = 0.4114$, and $A_3 = -0.2668$. Furthermore, the nondimensional value of the gas film force response at large time is $k(\infty) = 0.3152$, and the nondimensional mass is $m = 10^{-6}$. With these numbers the roots of the approximate characteristic equation can be found using a standard numerical algorithm. The roots are given in the first column of Table 1. The second and third roots have positive real parts; therefore, the step jump method using Laguerre polynomials indicates that the bearing is unstable. This result is clearly in contrast to the results discussed earlier and shown in Fig. 2 for this bearing.

The Laguerre coefficients and $k(\infty)$ are gas film properties and are independent of the mass of the bearing pad. Therefore, it is possible to study the effects of m on the bearing stability using classical methods such as Evans' root locus method (Franklin et al., 1994) and the Routh-Hurwitz stability criterion (Arnold and Maunder, 1961). For this example, the root locus for the parameter $1/m$ is shown in Figure 4. Two branches cross over from the left half plane (indicating stability) into the right half plane (indicating instability) at a critical value of $1/m_{\text{crit}} = 259.2$ or $m_{\text{crit}} = 0.00386$. This exact value for m_{crit} is also predicted by the Routh-Hurwitz stability criterion, where any $m < m_{\text{crit}}$ causes the system to become unstable. Note that in the above numerical example $m = 10^{-6}$ is indeed smaller than m_{crit}. Since the slider bearing investigated here must be physically stable for any mass m, it reaffirms the contradiction stated above.

Modeling the Gas Film as a Linear Viscoelastic Material

Several factors in the development of the step jump method show that a striking correspondence exists between the behavior of the gas film in triboelements and the characteristics of a linear viscoelastic material. To establish the correspondence, consider the force-displacement relationship for gas films, Eq. (4), in its dimensional form (with the star superscript added to signify dimensional quantities). Divide the equation through by the bearing area, $A_b = B \cdot 1$, and then multiply through by $h^*_{o,\text{eq}}/h^*_{o,\text{eq}}$, which yields

$$\sigma(t) = \epsilon(0)E(t) + \int_0^t \dot{\epsilon}(\tau)E(t-\tau)d\tau \quad (17)$$

with the following parameters defined,

$$\sigma(t) = \frac{f^*(t)}{B\cdot 1} \quad \epsilon(t) = \frac{y^*(t)}{h^*_{o,\text{eq}}} \quad E(t) = \frac{h^*_{o,\text{eq}}k^*(t)}{B\cdot 1} \quad (18)$$

Here, $\sigma(t)$ is the net force on the bearing pad per unit area, $\epsilon(t)$ is the nondimensional displacement, and $E(t)$ is the stiffness of the gas film per unit length. Equation (17) is exactly the stress-strain constitutive law for linear viscoelastic materials, which follows from the Boltzmann superposition principle (Gurtin and Sternberg, 1962). For the viscoelastic application, $E(t)$ is the relaxation modulus for the linear viscoelastic material. Typically, the relaxation modulus starts at a high value (the glassy modulus) at zero time and decays monotonically to a low value (the rubbery modulus) after a long time. An obvious result of this correlation is that the step response in the step jump method is equivalent to the relaxation modulus in viscoelasticity. Consequently, the step response is considered a relaxation function that represents the transient stiffness properties of a gas film in triboelement applications.

Further insight can be achieved by reexamining the time domain gas film constitutive law, Eq. (4). The equation can be reformulated as

$$f(t) = y(0)k(t) + \dot{y}(t)*k(t) \quad (19)$$

where the star product denotes the convolution of the functions $k(t)$ and $\dot{y}(t)$. The Laplace transform of Eq. (19) yields

$$F(s) = sK(s)Y(s) \quad (20)$$

Capital letter notation represents the Laplace transform of the corresponding time domain functions. In particular, $K(s)$ is the Laplace transform of the step response, $k(t)$.

Equation (20) suggests that a similar concept to the elastic-viscoelastic correspondence principle applies to gas lubricated triboelements. This principle states that a gas film problem can be formulated using first a pseudo linear spring to represent the stiffness of the fluid film:

Table 1 Roots of the characteristic equation, Eq. (16) or Eq. (27), for three nondimensional bearing pad masses

$m_1 = m = (10)^{-6}$	$m_2 = 3858.686 * m = m_{\text{crit}}$	$m_3 = (10)^4 * m$
$R_1 = -1.085 \pm i\ 0.0$	$R_1 = -1.095 \pm i\ 0.0$	$R_1 = -1.111 + i\ 0.0$
$R_2 = 0.892 + i\ 841.356$	$R_2 = 0.0 + i\ 14.417$	$R_2 = -0.464 + i\ 9.184$
$R_3 = 0.892 - i\ 841.356$	$R_3 = 0.0 - i\ 14.417$	$R_3 = -0.464 - i\ 9.184$
$R_4 = -3.705 + i\ 2.363$	$R_4 = -3.722 + i\ 2.392$	$R_4 = -3.757 + i\ 2.440$
$R_5 = -3.705 \pm i\ 2.363$	$R_5 = -3.722 - i\ 2.392$	$R_5 = -3.757 - i\ 2.440$
$R_6 = -13.890 + i\ 0.0$	$R_6 = -11.460 + i\ 0.0$	$R_6 = -10.448 + i\ 0.0$

$$f(t) = k_g y(t) \tag{21}$$

where k_g is the spring modulus. The appropriate force-displacement relationship is then achieved by replacing $f(t)$, $y(t)$, and k_g by $F(s)$, $Y(s)$, and $sK(s)$, respectively. As an example, consider again the gas lubricated slider bearing discussed earlier. The equation of motion for the bearing is given in Eq. (14). Applying the correspondence principle and specifically using Eq. (21), the equation of motion is rewritten as

$$m\ddot{y}(t) = -k_g y(t) \tag{22}$$

A Laplace domain version can be derived by making the appropriate substitutions, yielding

$$ms^2 Y(s) = -sK(s)Y(s) \tag{23}$$

The characteristic equation is now easily extracted:

$$ms^2 + sK(s) = 0 \tag{24}$$

Thermodynamic Compatibility of Response Functions. In viscoelastic applications, various analytic functions are commonly used to approximate the relaxation function in the viscoelastic constitutive law. Fabrizio and Morro (1988) developed a criterion that these analytic functions must satisfy to comply with the second law of thermodynamics. First, the criterion defines a loss factor, $K'(\omega)$, as

$$K'(\omega) = \int_0^\infty \dot{k}(t) \sin(\omega t) dt \quad \text{for all} \quad \omega \geq 0 \tag{25}$$

where $\dot{k}(t) = d[k(t)]/dt$. Next, the criterion states that the loss factor must be negative semi-definite and that the initial condition, $k(0)$, must be symmetric. Since the dynamic behavior of gas films and linear viscoelastic materials are mathematically equivalent, then any analytic function used to approximate the step response must also satisfy the same thermodynamic criterion.

Thermodynamic Inadmissibility of the Laguerre Polynomial Approximation. The method established above from Eq. (19) through Eq. (24) is thus far not compromised. For the case at hand, however, the analytic step response function is a series of Laguerre polynomials, Eq. (6), for which the resulting $sK(s)$ function is

$$sK(s) = k(\infty) + \sum_{n=0}^{N} A_n \left(\frac{s}{s+\alpha}\right)^{n+1} \tag{26}$$

Substituting Eq. (26) into Eq. (24) gives another form of the characteristic equation. For the case when $N = 3$, the approximate characteristic equation is

$$ms^2 + k(\infty) + A_0 \frac{s}{s+\alpha} + A_1 \left(\frac{s}{s+\alpha}\right)^2 + A_2 \left(\frac{s}{s+\alpha}\right)^3 + A_3 \left(\frac{s}{s+\alpha}\right)^4 = 0 \tag{27}$$

This equation derived using viscoelastic theory is exactly equivalent to Eq. (16), which was derived using the step jump method. Since the Laplace operator, s, and the complex variable, ν, are interchangeable, the same incorrect stability indications will be found, and the results will not be restated.

It is now possible to explain why the step jump method using Laguerre polynomials can give inaccurate stability results by subjecting the approximate function to the thermodynamic criterion outlined by Fabrizio and Morro (1988). The symmetry of the initial condition, $k(0)$, is guaranteed since $k(t)$ is one-dimensional. Using Eq. (6) in Eq. (25) gives the loss factor for the Laguerre polynomial series

$$K'(\omega) = \alpha \int_0^\infty \sum_{n=0}^{N} A_n L_n^1(\alpha t) \sin(\omega t) dt \quad \text{for all} \quad \omega \geq o \tag{28}$$

where L_n^1 is the first generalized Laguerre polynomial of the nth order. This expression does have an analytic solution, but its formulation is not easily generalized for an arbitrary value of N. However, for the present case $N = 3$, so

$$K'(\omega) = -\frac{(A_0 + 2A_1 + 3A_2 + 4A_3)\alpha\omega}{(\alpha^2 + \omega^2)} + \frac{(A_1 + 2A_2 + 6A_3)\alpha^2 \sin\left[2 \tan^{-1}\left(\frac{\omega}{\alpha}\right)\right]}{(\alpha^2 + \omega^2)} - \frac{(A_2 + 4A_3)\alpha^3 \sin\left[3 \tan^{-1}\left(\frac{\omega}{\alpha}\right)\right]}{(\alpha^2 + \omega^2)^{3/2}} + \frac{A_3 \alpha^4 \sin\left[4 \tan^{-1}\left(\frac{\omega}{\alpha}\right)\right]}{(\alpha^2 + \omega^2)^2} \tag{29}$$

N and α are the only two preselected variables in this equation. The polynomial coefficients, A_k, are curve fitting parameters

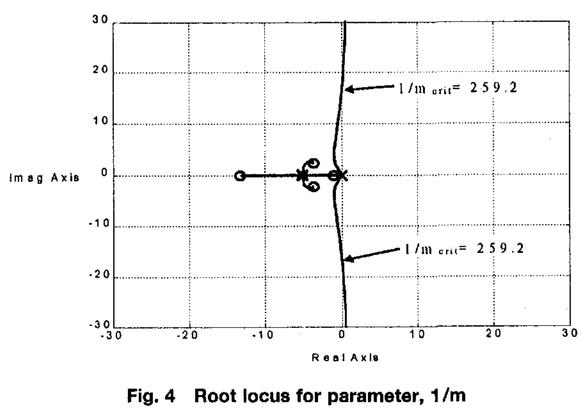

Fig. 4 Root locus for parameter, 1/m

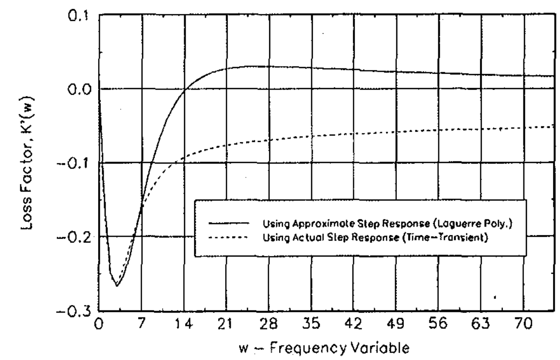

Fig. 5 Loss factor curves generated using approximate (Laguerre polynomials) and actual step responses

that depend on the shape of the step response. Therefore, it is very difficult to prove analytically that $K'(\omega)$ is negative semi-definite for an arbitrary set of bearing parameters. For a single set of data, however, it is sufficient to plot $K'(\omega)$ as a function of ω and from the plot, determine if the function is negative semi-definite.

The loss factor curves generated using the approximate and actual step responses are compared in Fig. 5. The curve corresponding to the approximate step response (see the solid line) was generated using Eq. (29) with the values of N, α, and A_k given earlier. The curve corresponding to the actual step response (see the dashed line) was obtained through numerical integration of Eq. (25) using the values obtained from Eq. (5). Since the loss factor for the actual step response is negative for all positive frequencies, the actual step response is shown to comply with the thermodynamic criterion. The curve generated using the approximate step response crosses over into positive values at a critical frequency and remains positive as the frequency increases. The fact the loss factor becomes positive proves that the step jump method with the Laguerre polynomial expansion contradicts the second law of thermodynamics at frequencies higher than the critical frequency. This is because the loss factor predicts whether energy will be dissipated out of or added into the system by the gas film at any frequency. When the natural frequency of the system is located in a region where the loss factor curve is negative, it implies that energy will be dissipated out of the system. However, the opposite is true for the case when the natural frequency is in a region where the loss factor is positive, thus implying that energy is added to the system.

To illustrate this phenomenon, consider the example stated earlier with the original mass, $m_1 = m = 10^{-6}$, and two variations: $m_2 = m_{\text{crit}} = 3859\ m$; and $m_3 = 10^4\ m$. These masses will result in three natural frequencies that fall into distinct regions of the loss factor curve, shown in Fig. 5, for the approximate step response. The roots of the characteristic equation, Eq. (16) or Eq. (27), for these three cases, are given in Table 1. The case in the third column corresponds to a system that is correctly predicted to be stable. The real parts of all the roots are negative, and the imaginary parts (i.e., the damped natural frequencies) are all within the region where the loss factor curve is negative. For the case in the first column, however, the real parts of the second and third roots are positive, and the system is predicted to be unstable. The imaginary parts of the second and third roots for this case correspond to a natural frequency that is in a region where the loss factor is positive. As a result, the model predicts that energy will be added to the system, thus creating an artificial instability. The second column is an interesting case where the real parts of the second and third roots are both zero. The natural frequencies for these roots are located at the threshold where the loss factor crosses the nondimensional ω axis at the critical frequency, $\omega_{\text{crit}} = 14.417$. For this case, the technique predicts that one mode will oscillate forever, which is again an incorrect estimate.

Viscous and squeeze effects in the gas film always result in energy being dissipated from the system. Since the gas film provides the only mechanism in the system through which energy is lost, the bearing is inherently stable. However, the Laguerre polynomial expansion model predicts that the gas film will add energy to the system if the natural frequency is above the critical frequency and, hence, adds an artificial (nonphysical) instability to the analysis. In general, compliance with the thermodynamic criterion is a necessary condition for stability. For the example analyzed here, that criterion is also sufficient to guarantee stability. (In different applications, say turbomachinery, there may be other factors that create instability, i.e., gyroscopic, or whirl effects.) Since the Laguerre polynomial expansion model can add artificial instabilities, there is no way to rely on the results of the step jump method using Laguerre polynomials without first verifying that it is thermodynamically admissible.

For the given slider bearing, there may be values for N and α for which the Laguerre polynomial expansion satisfies the thermodynamic criterion. In theory, those cases could be used as approximations for the step response. In general, however, for an analytic function to be considered as an approximation to the step response, it should be admissible for every set of system parameters that are physically possible. Otherwise, every case encountered during a dynamic analysis must be tested for thermodynamic compatibility, and that it is an unrealistic alternative.

Conclusions

The step jump approach is a widespread method used to determine the stability and to investigate parameter sensitivity in dynamic systems. However, it has been shown that the step jump method, as it is defined in literature, can give inaccurate results. The idea of obtaining the total response of the fluid layer by superposing the effects of individual step jumps is theoretically sound, but approximating the step jump response by a series of Laguerre polynomials is physically invalid. This conclusion was proven by considering the thermodynamics involved in the approach. An example was given for which the step jump method indicated that the system was unstable, but a conventional time-transient analysis proved otherwise. Although some cases exist where the Laguerre polynomial expansion technique predicts stability properly, this is not always the case. Consequently, it is concluded that the Laguerre polynomial expansion can lead to an invalid model of the dynamic properties of the gas film, and it should only be used if very careful attention is given to ensure that the second law of thermodynamics is not violated.

A significant conclusion of this work is that a correlation is shown to exist between the dynamic behavior of gas films in triboelements and the dynamic behavior of linear viscoelastic materials. This correlation is very advantageous because much of that theory can be applied to the dynamic analysis of triboelements. In accordance with this theory, there are many thermodynamically admissible analytic functions that have been used in viscoelastic applications (Szumski, 1993). Some examples are the complimentary error function, a Prony series, and Bessel series, and Bessel functions. These functions and others are legitimate possibilities for use in the dynamic analysis of triboelements using the step jump method.

Acknowledgment

This work was supported in part by an NSF Graduate Research Traineeship through Grant No. EEC-9256289. This support is gratefully acknowledged.

References

Arnold, R. N., and Maunder, L., 1961, *Gyrodynamics and its Engineering Applications*, Academic Press, New York.

Castelli, V., and McCabe, J. T., 1967, "Transient Dynamics of a Tilting Pad Gas Bearing System," ASME JOURNAL OF LUBRICATION TECHNOLOGY, Vol. 89, pp. 499–509.

Elrod, H. G., Jr., McCabe, J. T., and Chu, T. Y., 1967, "Determination of Gas-Bearing Stability by Response to a Step-Jump," ASME JOURNAL OF LUBRICATION TECHNOLOGY, Vol. 89, pp. 493–498.

Christensen, R. M., 1982, *Theory of Viscoelasticity*, Academic Press, New York.

Etsion, I., and Green, I., 1981, "Dynamic Analysis of a Cantilever-Mounted Gas-Lubricated Thrust Bearing," ASME JOURNAL OF LUBRICATION TECHNOLOGY, Vol. 103, pp. 157–163.

Fabrizio, M., and Morro, A., 1988, "Viscoelastic Relaxation Functions Compatible with Thermodynamics," *Journal of Elasticity*, Vol. 19, pp. 63–75.

Franklin, G. F., Powell, J. D., and Emami-Naeini, A., 1994, *Feedback Control of Dynamic Systems*, 3rd ed., Addison-Wesley, Reading, Massachusetts.

Gross, W. A., et al., 1980, *Fluid Film Lubrication*, Wiley, New York.

Gurtin, M. E., and Sternberg, E., 1962, "On the Linear Theory of Viscoelasticity," *Archive for Rational Mechanics and Analysis*, Vol. 11, pp. 291–356.

Kazimierski, Z., and Jarzecki, K., 1979, "Stability Threshold of Flexibly Supported Hybrid Gas Journal Bearings," ASME JOURNAL OF LUBRICATION TECHNOLOGY, Vol. 101, pp. 451–457.

Sela, N. M., and Blech, J. J., 1991, "Performance and Stability of a Hybrid Spherical Gas Gyrobearing," ASME JOURNAL OF TRIBOLOGY, Vol. 113, pp. 458–463.

Shapiro, W., and Colsher, R., 1970, "Implementation of Time-Transient and Step-Jump Dynamic Analyses of Gas-Lubricated Bearings," ASME JOURNAL OF LUBRICATION TECHNOLOGY, Vol. 92, pp. 518–529.

Szumski, R. G., 1993, "A Finite Element Formulation for the Time Domain Vibration Analysis of an Elastic-Viscoelastic Structure," Ph.D. thesis, Georgia Institute of Technology, Atlanta, GA.